Hello!

Logic and Computation:

A Constructive Look at Proofs of Gödel's Incompleteness Theorem*

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*A Joint Work with Payam Seraji.

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Why Constructivism?

Gregory J. CHAITIN, *Thinking about Gödel & Turing*, WS 2007, p. 97

So in the end it wasn't Gödel, it wasn't Turing, [...] that are making mathematics go in an experimental mathematics direction, in a quasi-empirical direction. The reason that mathematicians are changing their working habits is the computer. I think it's an excellent joke! (It's also funny that of the three old schools of mathematical philosophy, logicist, formalist, and intuitionist, the most neglected was BROUWER, who had a constructivist attitude years before the computer gave a tremendous impulse to constructivism.)

Why Constructive Proof(s)?

A Theorem with Constructive and Nonconstructive Proofs

A constructive (nonconstructive) proof shows the existence of an object by presenting (respectively, without presenting) the object. From a logical point of view, a constructive (nonconstructive) proof does not use (respectively, uses) the law of the excluded middle.

The discussion of constructive versus nonconstructive proofs is very common in mathematical logic and philosophy. To illustrate this discussion, it is convenient to have some *very simple* examples of theorems with both constructive and nonconstructive proofs. Unfortunately, there seems to be a shortage of such examples. We present here a new example.

Theorem. *Let c be an arbitrary real constant. The equation $c^2x^2 - (c^2 + c)x + c = 0$ in x has a real solution.*

Nonconstructive proof. By the law of the excluded middle, we have $c = 0$ or $c \neq 0$.

- Case $c = 0$: $x = 0$ (or any x) is a solution.
- Case $c \neq 0$: $x = 1/c$ is a solution.

(This proof is nonconstructive because it does not present a solution, that is, it does not decide between the two cases as the equality $c = 0$ is undecidable.) ■

The American Mathematical Monthly, vol. 120 no. 6 (2013) page 536.

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Constructive proof. We have that $x = 1$ is a solution. (This proof is constructive because it presents a solution.) ■

—Submitted by Jaime Gaspar,
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Constructive Proofs ⇝ Algorithms

Theorem (The Intermediate Value Theorem)

For any polynomial (in general, continuous) $f:\mathbb{R}\to\mathbb{R}$ if $f(a)f(b)<0$ then for some $c\in[a,b]$ we have $f(c)=0$.

Non-Constructive Proof.

Let $c=\sup\{x\in[a,b] : f(a)f(x)>0\}$ (the largest root of f in $[a,b]$) or $c=\inf\{x\in[a,b] : f(b)f(x)>0\}$ (the smallest). ❑

Constructive Proof.

Define $[a_n,b_n]$'s by induction: $[a_0,b_0]=[a,b]$, and

$$[a_{n+1},b_{n+1}]=\begin{cases}[a_n,\frac{a_n+b_n}{2}] & \text{if } f(a_n)f(\frac{a_n+b_n}{2})<0,\\ [\frac{a_n+b_n}{2},b_n] & \text{if } f(a_n)f(\frac{a_n+b_n}{2})>0,\\ \{\frac{a_n+b_n}{2}\} & \text{if } f(a_n)f(\frac{a_n+b_n}{2})=0;\end{cases}$$

and let $c=\lim_n a_n$ (or $\lim_n b_n$). ❑

Constructive Proofs $\rightsquigarrow$ Algorithms

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Another Example

Theorem (The Archemidean Property of the Rationals)

$\forall r \in \mathbb{Q}\, \exists n \in \mathbb{N} : r < n.$

Constructive Proof.

Write $r = \frac{p}{q}$ with $p \in \mathbb{Z}, q \in \mathbb{N}$. Now, from $1 \leqslant q$ we have $0 < \frac{1}{q} \leqslant 1$ and so $r = \frac{p}{q} \leqslant |p| < |p| + 1 (= n)$. ❑

Non-Constructive Proof.

If for $r = \frac{p}{q} \in \mathbb{Q}$, we have $\forall n \in \mathbb{N} : n \leqslant r$, then we can assume that $p, q \in \mathbb{N} - \{0\}$, and so $\frac{p}{q} > p$ whence $0 < q < 1$, contradiction! ❑

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The Most Well-Known Example (I)

Theorem (Some Irrational Power an Irrational Could Be Rational)

There are irrational numbers a, b such that a^b is rational.

Non-Constructive Proof.

If $\sqrt{2}^{\sqrt{2}}$ is rational then we are done with $a = b = \sqrt{2}$ (below) otherwise $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$ proves the theorem with $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$. ❑

Proof (of the irrationality of $\sqrt{2}$).

If $\sqrt{2} = \frac{p}{q}$ then $p^2 = 2q^2$, but the exponent of 2 in the unique prime factorization of p^2 is even while it is odd in $2q^2$, contradiction! ❑

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The Most Well-Known Example (II)

Theorem (Some Irrational Power an Irrational Could Be Rational)

There are irrational numbers a, b such that a^b is rational.

Constructive Proof.

For $a = \sqrt{2}, b = 2\log_2 3$ we have

$$a^b = (\sqrt{2})^{2\log_2 3} = 2^{\log_2 3} = 3.$$

❑

Proof (of the irrationality of $\log_2 3$).

If $\log_2 3 = \frac{p}{q}$ with $p, q \in \mathbb{N}-\{0\}$, then $q\log_2 3 = p$ and so $\log_2 3^q = p$ whence $3^q = 2^p$, contradiction! ❑

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The Most Well-Known Example (history)

- J. Roger Hindley: The Root-2 Proof as an Example of Non-Constructivity (March 2015, 3 pages)

  www.users.waitrose.com/~hindley/Root2Proof2015.pdf

- Dov Jarden, Curiosa No. 339, *Scripta Mathematica* 19 (1953) 229.
- Charles Zeigenfus, Quickie Q380, *Mathematics Magazine* 39 (1966) 134 (the question) 111 (the answer). $\sqrt{3}^{\sqrt{2}}$
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- George Pólya & Gabor Szegő, *Problems and Theorems in Analysis II*, Springer (1976) reprinted in 1998. $\sqrt{2}^{\log_2 9}$
- Joachim Lambek & Philip J. Scott, *Introduction to Higher-order Categorical Logic*, Cambridge University Press (1986). $\sqrt{2}^{\sqrt{2}}, \sqrt{2}^{\log_2 9}$
- Dirk van Dalen & Anne Sjerp Troelstra, *Constructivism in Mathematics*, Elsevier Science (1988).

Even More Constructive Proofs

A Constructive Proof for the irrationality of $\sqrt{2}$.

Since the parity of the exponents of 2 in p^2 and $2q^2$ are different (for any $p, q \in \mathbb{N}$), then $|2q^2 - p^2| \geqslant 1$. So, for any $0 < \frac{p}{q} < 3$ we have

$$|\sqrt{2} - \frac{p}{q}| = \frac{1}{q}|q\sqrt{2} - p| = \frac{|2q^2 - p^2|}{q(q\sqrt{2} + p)} \geqslant \frac{1}{q^2(\sqrt{2} + \frac{p}{q})} > (\frac{1}{2q})^2$$

because $\sqrt{2} + \frac{p}{q} < 1 + 3 = 4$. ❑

A Constructive Proof for the irrationality of $\log_2 3$.

¿¿¿———??? ❑

Some Computability (Recursion) Theory

Definition (Computably Decidable)

A set $A \subseteq \mathbb{N}$ with an algorithm $\mathcal{P}$ decides on any input x whether $x \in A$ (outputs YES) or $x \notin A$ (outputs NO).

$$\xrightarrow{\text{input: } x\in\mathbb{N}} \boxed{\text{Algorithm}} \xrightarrow{\text{output:}} \begin{cases} \texttt{YES} & \text{if } x \in A \\ \texttt{NO} & \text{if } x \notin A \end{cases}$$

Algorithm: single–input (natural number), Boolean–output (1/0). ✧

Definition (Semi–Decidable)

A set $A \subseteq \mathbb{N}$ with an algorithm $\mathcal{P}$ halts on any input x if and only if $x \in A$ (and does not halt if and only if $x \notin A$).

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Some More Computability Theory

Theorem (Decidability ≡ SemiDecidability + Co-SemiDecidability)

A set is decidable iff it and its complement are both semidecidable.

Proof.

If $\mathcal{P}$ semidecides A and $\mathcal{Q}$ semidecides $\overline{A}$ then for deciding A, on any input, run $\mathcal{P}$ and $\mathcal{Q}$ in parallel (a step of each in turn) and if $\mathcal{P}$ halts then print 1 and if $\mathcal{Q}$ halts then print 0. ❑

Convention (Classic Computability-Theoretic Notation)

Enumerate all the single-input computable (partial) functions $\mathbb{N}\rightarrow\mathbb{N}$ *as*

$$\varphi_0, \varphi_1, \varphi_2, \cdots .$$

Denote the universal (computable) function by $\Phi(x,y) = \varphi_x(y)$.

There exists a computable (partial) function $\Phi : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that for any computable (partial) function $f : \mathbb{N} \rightarrow \mathbb{N}$ there is some $e \in \mathbb{N}$ such that $f(x) = \Phi(e, x)$. ✧

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Computability Theory in Mathematical Logic

The set of PROOFS of an *Axiomatizable Theory* must be **Decidable**.

The **decidability** of its *set of axioms* suffices (and is necessary).

Proposition (Axioms $\in \Delta_1 \Longrightarrow$ Proofs $\in \Delta_1$ & Theorems $\in \Sigma_1$)

If the set of axioms of a theory is decidable, then the set of its proofs is decidable, and the set of its theorems is semi-decidable.

Proof.

If T is decidable, then the set of sequences $\langle \psi_0, \psi_1, \cdots, \psi_n \rangle$ with

- each ψ_i is either a logical axiom or a member of T, or
- each ψ_i results from some previous ones by an inference rule,

is decidable. Now, a formula ψ is a theorem of T if and only if one can find such a sequence with $\psi_n = \psi$. ❑

Proof Search Algorithm AUTOMATED THEOREM PROVING

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Proof Search Algorithm AUTOMATED THEOREM PROVING

A Semi-Decidable But Un-Decidable Set

Theorem (A Diagonal Argument)

There exists a semi-decidable but undecidable set.

	0	1	2	3	4	5	$\cdots$
φ_0	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$
φ_1	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\cdots$
φ_2	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\cdots$
φ_3	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\cdots$
φ_4	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\cdots$
φ_5	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\overline{K}$	X	1	2	3	X	X	$\cdots$
K	0	X	X	X	4	5	$\cdots$

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φ_3	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\cdots$
φ_4	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\cdots$
φ_5	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\cdots$
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Theorem (A Diagonal Argument)

There exists a semi-decidable but undecidable set.

(Constructive) Proof.

If $\overline{K} = \{n \in \mathbb{N} \mid \varphi_n(n)\uparrow\}$ were semi-decidable by (say) φ_k, then

$$\varphi_x(x)\uparrow \Longleftrightarrow x \in \overline{K} \Longleftrightarrow \varphi_k(x)\downarrow$$

so, for $x = k$,

$$\varphi_k(k)\uparrow \Longleftrightarrow \varphi_k(k)\downarrow,$$

contradiction!

Whence, $\overline{K}$, and also $K = \{n \in \mathbb{N} \mid \varphi_n(n)\downarrow\}$, is undecidable.

But the set $K = \{n \in \mathbb{N} \mid \varphi_n(n)\downarrow\}$ is semi-decidable by the (computable) function $n \mapsto \mathbf{\Phi}(n, n)$ since,

$$x \in K \longleftrightarrow \mathbf{\Phi}(x, x)\downarrow. \qquad \square$$

Gödel's First Incompleteness Theorem

Follows from (and in fact is equivalent to)
the existence of a semi-decidable but un-decidable set:

Theorem (Gödel's First Incompleteness Theorem—Semantic Form)

No semi-decidable and sound theory can be complete.

Kleene's Proof.

For a semi-decidable and undecidable set A (such that $\overline{A}$ is not semi-decidable) let $\overline{A}_T = \{n \in \mathbb{N} \mid T \vdash \text{“}n \notin A\text{”}\}$.
Then, by the soundness of T we have $\overline{A}_T \subseteq \overline{A}$,
but $\overline{A}_T$ is semi-decidable [$n \mapsto$ Proof-Search$_T(n \notin A)$] and $\overline{A}$ isn't.
So, there must be some $\mathbf{n} \in \overline{A}$ such that $\mathbf{n} \notin \overline{A}_T$.
Thus, $\mathbb{N} \models \mathbf{n} \notin A$ but $T \nvdash \text{“}\mathbf{n} \notin A\text{”}$. ❑

The proof in this form is not constructive, since
$\mathbf{n}$ is not (constructively) specified.

Gödel's First Incompleteness Theorem

Follows from (and in fact is equivalent to)
the existence of a semi-decidable but un-decidable set:

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Thus, $\mathbb{N} \models \mathbf{n} \notin A$ but $T \not\vdash \text{"}\mathbf{n} \notin A\text{"}$. ❑

The proof in this form is not constructive, since
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Gödel's First Incompleteness Theorem—Constructively

Kleene's Constructive Proof.

Let T be a sufficiently strong, sound and semi-decidable theory.

$$\{n\in\mathbb{N} \mid T\vdash \varphi_n(n)\uparrow\} \subseteq \{n\in\mathbb{N} \mid \varphi_n(n)\uparrow\}.$$

The first set is semi-decidable, say by
$\varphi_{\mathbf{t}}(x) = \text{Proof-Search}_T[\varphi_x(x)\uparrow]$, and the second set is not.

$$\varphi_{\mathbf{t}}(x)\downarrow \iff T\vdash \varphi_x(x)\uparrow$$

Now, on the one hand, (1) $\varphi_{\mathbf{t}}(\mathbf{t})\uparrow$, since otherwise (if $\varphi_{\mathbf{t}}(\mathbf{t})\downarrow$)

- ▷ by the sufficiently strongness of T, $T\vdash \varphi_{\mathbf{t}}(\mathbf{t})\downarrow$; and also
- ▷ $T\vdash \varphi_{\mathbf{t}}(\mathbf{t})\uparrow$; contradiction!

On the other hand, (2) $T\nvdash \varphi_{\mathbf{t}}(\mathbf{t})\uparrow$, since otherwise (if $T\vdash \varphi_{\mathbf{t}}(\mathbf{t})\uparrow$) we should had $\varphi_{\mathbf{t}}(\mathbf{t})\downarrow$, contradiction with (1)!

Thus, (1) $\varphi_{\mathbf{t}}(\mathbf{t})\uparrow$ and (2) $T\nvdash \varphi_{\mathbf{t}}(\mathbf{t})\uparrow$ (and also $T\nvdash \varphi_{\mathbf{t}}(\mathbf{t})\downarrow$). ❑

$$\varphi_{\mathbf{t}}(\mathbf{t})\downarrow \iff T\vdash \varphi_{\mathbf{t}}(\mathbf{t})\uparrow$$

Gödel's Proof

Gödel's Proof.

Denote the n-th Formula by $\mathcal{F}_n$ (via a Gödel coding).

$$\{n \in \mathbb{N} \mid T \vdash \neg\mathcal{F}_n(\overline{n})\} \subseteq \{n \in \mathbb{N} \mid \mathbb{N} \models \neg\mathcal{F}_n(\overline{n})\}.$$

The first set is arithmetically definable, while the second set is not!
(Tarski's Theorem: if it were by $\mathcal{F}_t(x)$ then $\mathcal{F}_t(t) \leftrightarrow \neg\mathcal{F}_t(t)$!).

The first set is definable by $\mathcal{F}_g(x)$; from $\mathcal{F}_g(x) \equiv T \vdash \neg\mathcal{F}_x(x)$ we have $\neg\mathcal{F}_g(g) \leftrightarrow T \nvdash \neg\mathcal{F}_g(g)$ (Gödel's Sentence).

So, for some sentence $\mathcal{G}$ we have $\mathcal{G} \equiv T \nvdash \mathcal{G}$ (Diagonal Lemma).

Now, $\mathbb{N} \models \mathcal{G}$, since otherwise $T \vdash \mathcal{G}$, and so $\mathbb{N} \models \mathcal{G}$.

Also, $T \nvdash \mathcal{G}$ since otherwise $\mathbb{N} \models \mathcal{G}$, contradiction!

Gödel's Paradox! ❑

Π_1-Incompleteness Theorems

Theorem (Proofs of the Uniform Π_1-Incompleteness Theorems)

Every uniform Π_1-incompleteness is of the form

$$\{n\in\mathbb{N} \mid T\vdash \text{“}n\notin\mathscr{A}\text{”}\} \subsetneqq \{n\in\mathbb{N} \mid \mathbb{N}\models \text{“}n\notin\mathscr{A}\text{”}\} = \overline{\mathscr{A}}$$

for some semi-decidable and un-decidable set $\mathscr{A}$.

If $\overline{\mathscr{A}}$ can be separated constructively from its every semi-decidable subset, then the proof is constructive; otherwise non-constructive.

Definition (Creative)

A set A is creative, if it is semi-decidable and there exists a (partial) computable function $f:\mathbb{N}\to\mathbb{N}$ such that for every n, if B is a subset of $\overline{A}$ which is semi-decidable by φ_n, then $f(n)\in\overline{A}-B$. ✧

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Creative and Non-Creative Semi-Decidability

Emil L. Post, *Recursively Enumerable Sets of Positive Integers and their Decision Problems*, Bulletin AMS 50 (1944) p. 295.

"… every symbolic logic is incomplete […]. The conclusion is unescapable that even for such a fixed, well defined body of mathematical propositions, *mathematical thinking is, and must remain, essentially creative*."

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Chaitin-Kolmogorov Complexity

Definition (Information-Theoretic Complexity)

For any $n \in \mathbb{N}$, the complexity of n is defined to be the least *size* of a program that generates (outputs) n (without specifying an input). ✧

Definition (Kolmogorov Complexity)

$\mathscr{K}(n) = \min\{m \mid \varphi_m(0) = n\}$. ✧

Lemma (The Main Lemma on the Kolmogorov Complexity)

For any m there is some ℓ such that $\mathscr{K}(\ell) > m$.

Non-Constructive Proof.

There are at most $m+1$ values for $\varphi_0(0), \varphi_1(0), \cdots, \varphi_m(0)$; so any number ℓ not from this list satisfies $\mathscr{K}(\ell) > m$. ❑

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Non-Constructive Theorems / Proofs

Theorem (Non-Constructivity of the Main Lemma)

There is no computable function f such that $\forall m : \mathscr{K}\big(f(m)\big) > m$.

Berry Paradox:

The Smallest Number Not Outputable by Program-*Size* of $\leqslant \cdots$

Proof.

For any such f, let $g(x)$ be a code for the constant function $n \mapsto f(x)$. By Kleene's fixed point theorem there exists some e such that $\varphi_e(n) = \varphi_{g(e)}(n) = f(e)$. So, in particular, $\varphi_e(0) = f(e)$, thus $\mathscr{K}\big(f(e)\big) \leqslant e$, contradiction! ❑

So, the Main Lemma on the Kolmogorov Complexity is (essentially) non-constructive, with a constructive proof! For any φ_k one can constructively find some e_k such that $\mathscr{K}\big(\varphi_k(e_k)\big) \leqslant e_k$.

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Chaitin's Incompleteness Theorem

Theorem (Chaitin's Theorem)

For any sound and semi-decidable theory there are w, m such that $\mathscr{K}(w) > m$ but the theory cannot prove that.

Non-Constructive Proof.

For any such T there is some m such that $T \nvdash \mathscr{K}(\omega) > m$ for any ω.
Since, otherwise if for any m there were some ω such that $T \vdash \mathscr{K}(\omega) > m$ then, for a given m, by a proof-search algorithm one could constructively find some ω with $(T \vdash)\mathscr{K}(\omega) > m$ contradicting the non-constructivity of the Main Lemma.
For a fixed such an m, by the Main Lemma, there is some w with $\mathscr{K}(w) > m$; and of course $T \nvdash \mathscr{K}(w) > m$. ❑

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Non-Constructivity of Chaitin's Proof

Theorem (Proof Idea from Denis R. Hirschfeldt)

The set $\{\langle w, m\rangle \mid \mathscr{K}(w) \leqslant m\}$ *is not creative.*

http://mathoverflow.net/questions/222925/ 7–10 Nov. 2015

Gregory J. Chaitin, A Century of Controversy Over the Foundations of Mathematics, *Complexity* 5 (2000) p. 21

> But I must say that philosophers have not picked up the ball. I think logicians hate my work, they detest it! And I'm like pornography, I'm sort of an unmentionable subject in the world of logic, because my results are so disgusting!
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Thank You!

Thanks to

The Participants .... For Listening ···

and

The Organizers — For Taking Care of Everything ···

SaeedSalehi.ir