

*Hello!*

## Logic and Computation:

A Constructive Look at Proofs of Gödel's Incompleteness Theorem\*

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## Why Constructivism?

Gregory J. CHAITIN, *Thinking about Gödel & Turing*, WS 2007, p. 97

So in the end it wasn't Gödel, it wasn't Turing, [...] that are making mathematics go in an experimental mathematics direction, in a quasi-empirical direction. The reason that mathematicians are changing their working habits is the **computer**. I think it's an excellent joke! (It's also funny that of the three old schools of mathematical philosophy, **logicist**, **formalist**, and **intuitionist**, the most neglected was **BROUWER**, who had a constructivist attitude years before the computer gave a tremendous impulse to **constructivism**.)

# Why Constructive Proof(s)?

## A Theorem with Constructive and Nonconstructive Proofs

A constructive (nonconstructive) proof shows the existence of an object by presenting (respectively, without presenting) the object. From a logical point of view, a constructive (nonconstructive) proof does not use (respectively, uses) the law of the excluded middle.

The discussion of constructive versus nonconstructive proofs is very common in mathematical logic and philosophy. To illustrate this discussion, it is convenient to have some *very simple* examples of theorems with both constructive and nonconstructive proofs. Unfortunately, there seems to be a shortage of such examples. We present here a new example.

**Theorem.** *Let  $c$  be an arbitrary real constant. The equation  $c^2x^2 - (c^2 + c)x + c = 0$  in  $x$  has a real solution.*

*Nonconstructive proof.* By the law of the excluded middle, we have  $c = 0$  or  $c \neq 0$ .

- Case  $c = 0$ :  $x = 0$  (or any  $x$ ) is a solution.
- Case  $c \neq 0$ :  $x = 1/c$  is a solution.

(This proof is nonconstructive because it does not present a solution, that is, it does not decide between the two cases as the equality  $c = 0$  is undecidable.) ■

*The American Mathematical Monthly*, vol. 120 no. 6 (2013) page 536.

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*Constructive proof.* We have that  $x = 1$  is a solution. (This proof is constructive because it presents a solution.) ■

—Submitted by Jaime Gaspar,  
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# Constructive Proofs $\rightsquigarrow$ Algorithms

## Theorem (The Intermediate Value Theorem)

For any polynomial (in general, continuous)  $f: \mathbb{R} \rightarrow \mathbb{R}$  if  $f(a)f(b) < 0$  then for some  $c \in [a, b]$  we have  $f(c) = 0$ .

### Non-Constructive Proof.

Let  $c = \sup \{x \in [a, b] : f(a)f(x) > 0\}$  (the largest root of  $f$  in  $[a, b]$ ) or  $c = \inf \{x \in [a, b] : f(b)f(x) > 0\}$  (the smallest).  $\square$

### Constructive Proof.

Define  $[a_n, b_n]$ 's by induction:  $[a_0, b_0] = [a, b]$ , and

$$[a_{n+1}, b_{n+1}] = \begin{cases} [a_n, \frac{a_n+b_n}{2}] & \text{if } f(a_n)f(\frac{a_n+b_n}{2}) < 0, \\ [\frac{a_n+b_n}{2}, b_n] & \text{if } f(a_n)f(\frac{a_n+b_n}{2}) > 0, \\ \{\frac{a_n+b_n}{2}\} & \text{if } f(a_n)f(\frac{a_n+b_n}{2}) = 0; \end{cases}$$

and let  $c = \lim_n a_n$  (or  $\lim_n b_n$ ).  $\square$

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## Another Example

### Theorem (The Archimedean Property of the Rationals)

$$\forall r \in \mathbb{Q} \exists n \in \mathbb{N} : r < n.$$

#### Constructive Proof.

Write  $r = \frac{p}{q}$  with  $p \in \mathbb{Z}, q \in \mathbb{N}$ . Now, from  $1 \leq q$  we have  $0 < \frac{1}{q} \leq 1$  and so  $r = \frac{p}{q} \leq |p| < |p| + 1 (= n)$ .  $\square$

#### Non-Constructive Proof.

If for  $r = \frac{p}{q} \in \mathbb{Q}$ , we have  $\forall n \in \mathbb{N} : n \leq r$ , then we can assume that  $p, q \in \mathbb{N} - \{0\}$ , and so  $\frac{p}{q} > p$  whence  $0 < q < 1$ , contradiction!  $\square$



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# The Most Well-Known Example (I)

Theorem (Some Irrational Power an Irrational Could Be Rational)

*There are irrational numbers  $a, b$  such that  $a^b$  is rational.*

Non-Constructive Proof.

If  $\sqrt{2}^{\sqrt{2}}$  is rational then we are done with  $a = b = \sqrt{2}$  (below)  
 otherwise  $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$  proves the theorem with  
 $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$ . □

Proof (of the irrationality of  $\sqrt{2}$ ).

If  $\sqrt{2} = \frac{p}{q}$  then  $p^2 = 2q^2$ , but the exponent of 2 in the unique prime factorization of  $p^2$  is even while it is odd in  $2q^2$ , contradiction! □

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Theorem (Some Irrational Power an Irrational Could Be Rational)

*There are irrational numbers  $a, b$  such that  $a^b$  is rational.*

Constructive Proof.

For  $a = \sqrt{2}, b = 2 \log_2 3$  we have

$$a^b = (\sqrt{2})^{2 \log_2 3} = 2^{\log_2 3} = 3. \quad \square$$

Proof (of the irrationality of  $\log_2 3$ ).

If  $\log_2 3 = \frac{p}{q}$  with  $p, q \in \mathbb{N} - \{0\}$ , then  $q \log_2 3 = p$  and so  $\log_2 3^q = p$  whence  $3^q = 2^p$ , contradiction! □

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# The Most Well-Known Example (history)

## ► J. ROGER HINDLEY: The Root-2 Proof as an Example of Non-Constructivity (March 2015, 3 pages)

[www.users.waitrose.com/~hindley/Root2Proof2015.pdf](http://www.users.waitrose.com/~hindley/Root2Proof2015.pdf)

- DOV JARDEN, Curiosa No. 339, *Scripta Mathematica* 19 (1953) 229.
- CHARLES ZEIGENFUS, Quickie Q380, *Mathematics Magazine* 39 (1966) 134 (the question) 111 (the answer).  $\sqrt{3}^{\sqrt{2}}$
- DIRK VAN DALEN, “Lectures on Intuitionism”, in: *Cambridge Summer School in Mathematical Logic*, (UK, Aug. 1971); Springer, LNM 337 (1973) pp. 1–94.
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- GEORGE PÓLYA & GABOR SZEGŐ, *Problems and Theorems in Analysis II*, Springer (1976) reprinted in 1998.  $\sqrt{2}^{\log_2 9}$
- JOACHIM LAMBEK & PHILIP J. SCOTT, *Introduction to Higher-order Categorical Logic*, Cambridge University Press (1986).  $\sqrt{2}^{\sqrt{2}}, \sqrt{2}^{\log_2 9}$
- DIRK VAN DALEN & ANNE SJERP TROELSTRA, *Constructivism in Mathematics*, Elsevier Science (1988).

## Even More Constructive Proofs

### A Constructive Proof for the irrationality of $\sqrt{2}$ .

Since the parity of the exponents of 2 in  $p^2$  and  $2q^2$  are different (for any  $p, q \in \mathbb{N}$ ), then  $|2q^2 - p^2| \geq 1$ . So, for any  $0 < \frac{p}{q} < 3$  we have

$$\left| \sqrt{2} - \frac{p}{q} \right| = \frac{1}{q} |q\sqrt{2} - p| = \frac{|2q^2 - p^2|}{q(q\sqrt{2} + p)} \geq \frac{1}{q^2(\sqrt{2} + \frac{p}{q})} > \left(\frac{1}{2q}\right)^2$$

because  $\sqrt{2} + \frac{p}{q} < 1 + 3 = 4$ . □

### A Constructive Proof for the irrationality of $\log_2 3$ .

??? ——— ??? □

# Some Computability (Recursion) Theory

## Definition (Computationally Decidable)

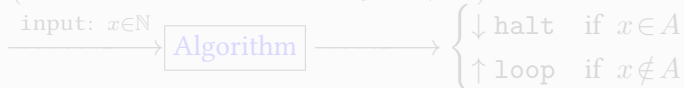
A set  $A \subseteq \mathbb{N}$  with an algorithm  $\mathcal{P}$  decides on any input  $x$  whether  $x \in A$  (outputs YES) or  $x \notin A$  (outputs NO).



**Algorithm:** single-input (natural number), Boolean-output (1/0).  $\diamond$

## Definition (Semi-Decidable)

A set  $A \subseteq \mathbb{N}$  with an algorithm  $\mathcal{P}$  halts on any input  $x$  if and only if  $x \in A$  ( and does not halt if and only if  $x \notin A$  ).

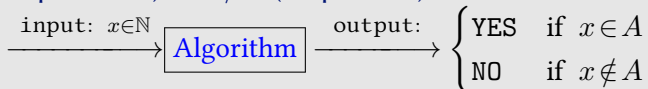


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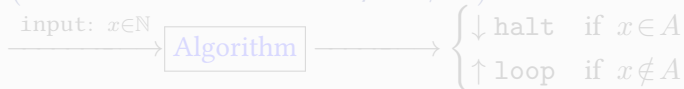
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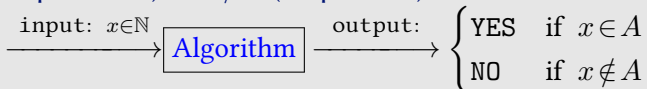


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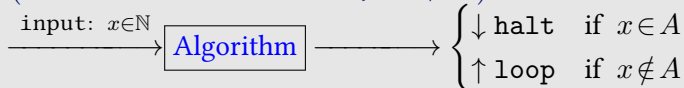
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## Some More Computability Theory

Theorem (**Decidability  $\equiv$  SemiDecidability + Co-SemiDecidability**)

*A set is decidable iff it and its complement are both semidecidable.*

Proof.

If  $\mathcal{P}$  semidecides  $A$  and  $\mathcal{Q}$  semidecides  $\bar{A}$  then for deciding  $A$ , on any input, run  $\mathcal{P}$  and  $\mathcal{Q}$  in parallel (a step of each in turn) and if  $\mathcal{P}$  halts then print 1 and if  $\mathcal{Q}$  halts then print 0.  $\square$

Convention (**Classic Computability-Theoretic Notation**)

*Enumerate all the single-input computable (partial) functions  $\mathbb{N} \rightarrow \mathbb{N}$  as*

$$\varphi_0, \varphi_1, \varphi_2, \dots$$

*Denote the universal (computable) function by  $\Phi(x, y) = \varphi_x(y)$ .*

*There exists a computable (partial) function  $\Phi: \mathbb{N}^2 \rightarrow \mathbb{N}$  such that for any computable (partial) function  $f: \mathbb{N} \rightarrow \mathbb{N}$  there is some  $e \in \mathbb{N}$  such that  $f(x) = \Phi(e, x)$ .*  $\diamond$

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# Computability Theory in Mathematical Logic

The set of PROOFS of an *Axiomatizable Theory* must be **Decidable**.

The **decidability** of its *set of axioms* suffices (and is necessary).

Proposition ( $\text{Axioms} \in \Delta_1 \implies \text{Proofs} \in \Delta_1 \ \& \ \text{Theorems} \in \Sigma_1$ )

*If the set of axioms of a theory is decidable, then the set of its proofs is decidable, and the set of its theorems is semi-decidable.*

Proof.

If  $T$  is decidable, then the set of sequences  $\langle \psi_0, \psi_1, \dots, \psi_n \rangle$  with

- each  $\psi_i$  is either a logical axiom or a member of  $T$ , or
- each  $\psi_i$  results from some previous ones by an inference rule,

is decidable. Now, a formula  $\psi$  is a theorem of  $T$  if and only if one can find such a sequence with  $\psi_n = \psi$ . □

Proof Search Algorithm

AUTOMATED THEOREM PROVING



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# A Semi-Decidable But Un-Decidable Set

## Theorem (A Diagonal Argument)

*There exists a semi-decidable but undecidable set.*

	0	1	2	3	4	5	...
$\varphi_0$	↓	↓	↓	↓	↓	↓	...
$\varphi_1$	↓	↑	↓	↑	↓	↑	...
$\varphi_2$	↑	↑	↑	↑	↑	↑	...
$\varphi_3$	↑	↑	↑	↑	↓	↓	...
$\varphi_4$	↓	↓	↑	↑	↓	↓	...
$\varphi_5$	↑	↓	↓	↓	↑	↓	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\bar{K}$	X	1	2	3	X	X	...
$K$	0	X	X	X	4	5	...

# A Semi-Decidable But Un-Decidable Set

## Theorem (A Diagonal Argument)

*There exists a semi-decidable but undecidable set.*

	0	1	2	3	4	5	...
$\varphi_0$	↓	↓	↓	↓	↓	↓	...
$\varphi_1$	↓	↑	↓	↑	↓	↑	...
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# A Semi-Decidable But Un-Decidable Set

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*There exists a semi-decidable but undecidable set.*

(Constructive) Proof.

If  $\overline{K} = \{n \in \mathbb{N} \mid \varphi_n(n) \uparrow\}$  were semi-decidable by (say)  $\varphi_k$ , then

so, for  $x = k$ , 
$$\varphi_x(x) \uparrow \iff x \in \overline{K} \iff \varphi_k(x) \downarrow$$

contradiction! 
$$\varphi_k(k) \uparrow \iff \varphi_k(k) \downarrow,$$

Whence,  $\overline{K}$ , and also  $K = \{n \in \mathbb{N} \mid \varphi_n(n) \downarrow\}$ , is undecidable.

But the set  $K = \{n \in \mathbb{N} \mid \varphi_n(n) \downarrow\}$  is semi-decidable by the (computable) function  $n \mapsto \Phi(n, n)$  since,

$$x \in K \iff \Phi(x, x) \downarrow.$$



# Gödel's First Incompleteness Theorem

Follows from (and in fact is equivalent to)  
the existence of a semi-decidable but un-decidable set:

Theorem (Gödel's First Incompleteness Theorem—Semantic Form)

*No semi-decidable and sound theory can be complete.*

Kleene's Proof.

For a semi-decidable and undecidable set  $A$  (such that  $\bar{A}$  is not semi-decidable) let  $\bar{A}_T = \{n \in \mathbb{N} \mid T \vdash "n \notin A"\}$ .

Then, by the soundness of  $T$  we have  $\bar{A}_T \subseteq \bar{A}$ ,  
but  $\bar{A}_T$  is semi-decidable [ $n \mapsto \text{Proof-Search}_T(n \notin A)$ ] and  $\bar{A}$  isn't.  
So, there must be some  $n \in \bar{A}$  such that  $n \notin \bar{A}_T$ .

Thus,  $\mathbb{N} \models n \notin A$  but  $T \not\vdash "n \notin A"$ . □

The proof in this form is not constructive, since  
 $n$  is not (constructively) specified.

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Thus,  $\mathbb{N} \models \mathbf{n} \notin A$  but  $T \not\vdash "\mathbf{n} \notin A"$ . □

The proof in this form is not constructive, since

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# Gödel's First Incompleteness Theorem—Constructively

## Kleene's Constructive Proof.

Let  $T$  be a sufficiently strong, sound and semi-decidable theory.

$$\{n \in \mathbb{N} \mid T \vdash \varphi_n(n) \uparrow\} \subseteq \{n \in \mathbb{N} \mid \varphi_n(n) \uparrow\}.$$

The first set is semi-decidable, say by

$\varphi_t(x) = \text{Proof-Search}_T[\varphi_x(x) \uparrow]$ , and the second set is not.

$$\varphi_t(x) \downarrow \iff T \vdash \varphi_x(x) \uparrow$$

Now, on the one hand, (1)  $\varphi_t(t) \uparrow$ , since otherwise (if  $\varphi_t(t) \downarrow$ )

▷ by the sufficient strongness of  $T$ ,  $T \vdash \varphi_t(t) \downarrow$ ; and also

▷  $T \vdash \varphi_t(t) \uparrow$ ; contradiction!

On the other hand, (2)  $T \not\vdash \varphi_t(t) \uparrow$ , since otherwise (if  $T \vdash \varphi_t(t) \uparrow$ ) we should had  $\varphi_t(t) \downarrow$ , contradiction with (1)!

Thus, (1)  $\varphi_t(t) \uparrow$  and (2)  $T \not\vdash \varphi_t(t) \uparrow$  (and also  $T \not\vdash \varphi_t(t) \downarrow$ ). □

$$\varphi_t(t) \downarrow \iff T \vdash \varphi_t(t) \uparrow$$

# Gödel's Proof

## Gödel's Proof.

Denote the  $n$ -th Formula by  $\mathcal{F}_n$  (via a Gödel coding).

$$\{n \in \mathbb{N} \mid T \vdash \neg \mathcal{F}_n(\bar{n})\} \subseteq \{n \in \mathbb{N} \mid \mathbb{N} \models \neg \mathcal{F}_n(\bar{n})\}.$$

The first set is arithmetically definable, while the second set is not!

(**Tarski's Theorem**: if it were by  $\mathcal{F}_t(x)$  then  $\mathcal{F}_t(t) \leftrightarrow \neg \mathcal{F}_t(t)$ !).

The first set is definable by  $\mathcal{F}_g(x)$ ; from  $\mathcal{F}_g(x) \equiv T \vdash \neg \mathcal{F}_x(x)$  we have  $\neg \mathcal{F}_g(g) \leftrightarrow T \not\vdash \neg \mathcal{F}_g(g)$  (**Gödel's Sentence**).

So, for some sentence  $\mathcal{G}$  we have  $\mathcal{G} \equiv T \not\vdash \mathcal{G}$  (Diagonal Lemma).

Now,  $\mathbb{N} \models \mathcal{G}$ , since otherwise  $T \vdash \mathcal{G}$ , and so  $\mathbb{N} \models \mathcal{G}$ .

Also,  $T \not\vdash \mathcal{G}$  since otherwise  $\mathbb{N} \models \mathcal{G}$ , contradiction!

**Gödel's Paradox!**



## $\Pi_1$ -Incompleteness Theorems

Theorem (Proofs of the Uniform  $\Pi_1$ -Incompleteness Theorems)

Every uniform  $\Pi_1$ -incompleteness is of the form

$$\{n \in \mathbb{N} \mid T \vdash "n \notin \mathcal{A}"\} \subsetneq \{n \in \mathbb{N} \mid \mathbb{N} \models "n \notin \mathcal{A}"\} = \overline{\mathcal{A}}$$

for some semi-decidable and un-decidable set  $\mathcal{A}$ .

If  $\overline{\mathcal{A}}$  can be separated constructively from its every semi-decidable subset, then the proof is constructive; otherwise non-constructive.

Definition (Creative)

A set  $A$  is creative, if it is semi-decidable and there exists a (partial) computable function  $f: \mathbb{N} \rightarrow \mathbb{N}$  such that for every  $n$ , if  $B$  is a subset of  $\overline{A}$  which is semi-decidable by  $\varphi_n$ , then  $f(n) \in \overline{A} - B$ .  $\diamond$

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## Creative and Non-Creative Semi-Decidability

Emil L. POST, *Recursively Enumerable Sets of Positive Integers and their Decision Problems*, Bulletin AMS 50 (1944) p. 295.

“... every symbolic logic is incomplete [...]. The conclusion is unescapable that even for such a fixed, well defined body of mathematical propositions, *mathematical thinking is, and must remain, essentially creative.*”

Martin D. DAVIS, *What is a Computation?*, in: *Mathematics Today*, twelve informal essays (ed. L.A. Steen, Springer 1978) p. 265; and in: *Randomness and Complexity, from Leibniz to Chaitin* (ed. C.S. Calude, WS 2007) p. 110.

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# Chaitin-Kolmogorov Complexity

## Definition (Information-Theoretic Complexity)

For any  $n \in \mathbb{N}$ , the **COMPLEXITY** of  $n$  is defined to be the least *size* of a **program** that generates (outputs)  $n$  (without specifying an input).  $\diamond$

## Definition (Kolmogorov Complexity)

$$\mathcal{K}(n) = \min \{m \mid \varphi_m(0) = n\}.$$

 $\diamond$ 

## Lemma (The Main Lemma on the Kolmogorov Complexity)

*For any  $m$  there is some  $\ell$  such that  $\mathcal{K}(\ell) > m$ .*

## Non-Constructive Proof.

There are at most  $m + 1$  values for  $\varphi_0(0), \varphi_1(0), \dots, \varphi_m(0)$ ; so any number  $\ell$  not from this list satisfies  $\mathcal{K}(\ell) > m$ .  $\square$



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# Non-Constructive Theorems / Proofs

## Theorem (Non-Constructivity of the Main Lemma)

*There is no computable function  $f$  such that  $\forall m : \mathcal{K}(f(m)) > m$ .*

### BERRY PARADOX:

The Smallest Number Not Outputable by Program-Size of  $\leq \dots$

### Proof.

For any such  $f$ , let  $g(x)$  be a code for the constant function  $n \mapsto f(x)$ . By Kleene's fixed point theorem there exists some  $e$  such that  $\varphi_e(n) = \varphi_{g(e)}(n) = f(e)$ . So, in particular,  $\varphi_e(0) = f(e)$ , thus  $\mathcal{K}(f(e)) \leq e$ , contradiction!  $\square$

So, the Main Lemma on the Kolmogorov Complexity is (essentially) non-constructive, with a constructive proof! For any  $\varphi_k$  one can constructively find some  $e_k$  such that  $\mathcal{K}(\varphi_k(e_k)) \leq e_k$ .

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# Chaitin's Incompleteness Theorem

## Theorem (Chaitin's Theorem)

*For any sound and semi-decidable theory there are  $w, m$  such that  $\mathcal{K}(w) > m$  but the theory cannot prove that.*

## Non-Constructive Proof.

For any such  $T$  there is some  $m$  such that  $T \not\vdash \mathcal{K}(w) > m$  for any  $w$ . Since, otherwise if for any  $m$  there were some  $w$  such that  $T \vdash \mathcal{K}(w) > m$  then, for a given  $m$ , by a proof-search algorithm one could constructively find some  $w$  with  $(T \vdash) \mathcal{K}(w) > m$  contradicting the non-constructivity of the Main Lemma. For a fixed such an  $m$ , by the Main Lemma, there is some  $w$  with  $\mathcal{K}(w) > m$ ; and of course  $T \not\vdash \mathcal{K}(w) > m$ . □

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*The set  $\{\langle w, m \rangle \mid \mathcal{K}(w) \leq m\}$  is not creative.*

<http://mathoverflow.net/questions/222925/> 7–10 Nov. 2015

Gregory J. CHAITIN, A Century of Controversy Over the Foundations of Mathematics, *Complexity* 5 (2000) p. 21

But I must say that philosophers have not picked up the ball. I think logicians hate my work, they detest it! And I'm like pornography, I'm sort of an unmentionable subject in the world of logic, because my results are so disgusting!

... the most interesting thing about the field of program-size complexity is that it has no applications, is that it proves that it cannot be applied! Because you can't calculate the size of the smallest program. But that's what's fascinating about it, because it reveals limits to what we can know. That's why program-size complexity has epistemological significance.

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Thank You!

Thanks to

The Participants ..... For Listening ...

and

The Organizers — For Taking Care of Everything ...

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