Modal Logics (Normal & Non-Normal)

Saeed Salehi

University of Tabriz

http://SaeedSalehi.ir/

Modal Logic and Computer Science

June 3′ 2010 ‖ IPM, Tehran

Modality Operator

Philosophy – Logic – Computer Science

$$\Box A$$

Necessity – Provability – Program Execution

$$\Box A \longrightarrow A$$

Philosophy:	`Necessity implies Truth`
Logic:	`Provability implies Validity`
Computer Science:	`Program is Sound`

Modality Operator

Philosophy – Logic – Computer Science

$$\Box A$$

Necessity – Provability – Program Execution

$$\Box A \longrightarrow \Box\Box A$$

Philosophy:	"Necessity" is Necessary
Logic:	"Provability" is Provable
Computer Science:	"Executability" is Executable

Modality Operator

Philosophy – Logic – Computer Science

$$\Box A$$

Necessity – Provability – Program Execution

$$\Box A \longrightarrow A \qquad A := \bot \qquad \boxed{\neg\Box\bot}$$

Philosophy: `Falsity is Not Necessary`
Logic: `Contradiction is Not Provable` (CONSISTENT)
Computer Science: `Program does Not go Absurd`

Other Modality Operators

Philosophy – Logic – Computer Science

$$\Diamond A$$

Possibility – Consistency – Probable Result

Define $\Diamond A = \neg\Box\neg A$ (So $\Box A = \neg\Diamond\neg A$)

$$\Diamond\Diamond A \longrightarrow \Diamond A \quad \text{or} \quad \Box A \longrightarrow \Box\Box A$$

Philosophy: `If the Possibility of A is Possible, then A is indeed Possible`

Other Modality Operators

Philosophy – Logic – Computer Science

$$\Diamond A$$

Possibility – Consistency – Probable Result

Define $\Diamond A = \neg\Box\neg A$ (So $\Box A = \neg\Diamond\neg A$)

$$\Diamond\Diamond A \longrightarrow \Diamond A \quad \text{or} \quad \Box A \longrightarrow \Box\Box A$$

Logic: `If the Consistency of A is Consistent, then A is consistent`

Propositional Modal Logics

Classical Propositional Calculus +
Modality Axioms and Rules

Axiom: $(\mathtt{K})\ \Box(A \to B) \to (\Box A \to \Box B)$
Rule:

$$(\mathtt{RN})\ \frac{A}{\Box A}$$

This base logic is denoted $\mathbf{K}$.

Normal Modal Logics $\supseteq \mathbf{K}$

Propositional Modal Logics

Adding more axioms ⇉ stronger modal logics:

(T) $\Box A \to A$ (3)

(4) $\Box A \to \Box\Box A$

(5) $\Box A \to \Box\Diamond A$

(B) $A \to \Box\Diamond A$

(D) $\Box A \to \Diamond A$

(L) $\Box(\Box A \to A) \to \Box A$

(K) $\Box(A \to B) \to (\Box A \to \Box B)$ (2)

($\Diamond$) $\neg\Diamond A \leftrightarrow \Box\neg A$ (1)

$$(\mathtt{K}) + (\mathtt{L}) + (\mathtt{RN}) = \mathbf{GL} \vdash (4).$$

Modal Logic

First published Tue Feb 29, 2000; substantive revision Fri Oct 2, 2009

A modal is an expression (like 'necessarily' or 'possibly') that is used to qualify the truth of a judgement. Modal logic is, strictly speaking, the study of the deductive behavior of the expressions 'it is necessary that' and 'it is possible that'. However, the term 'modal logic' may be used more broadly for a family of related systems. These include logics for belief, for tense and other temporal expressions, for the deontic (moral) expressions such as 'it is obligatory that' and 'it is permitted that', and many others. An understanding of modal logic is particularly valuable in the formal analysis of philosophical argument, where expressions from the modal family are both common and confusing. Modal logic also has important applications in computer science.

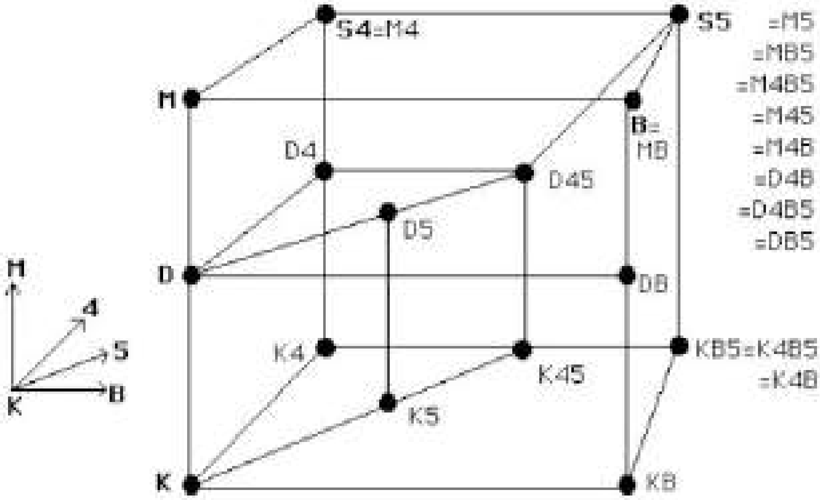
---

1. What is Modal Logic?

Narrowly construed, modal logic studies reasoning that involves the use of the expressions 'necessarily' and 'possibly'. However, the term 'modal logic' is used more broadly to cover a family of logics with similar rules and a variety of different symbols.

8. Map of the Relationships Between Modal Logics

The following diagram shows the relationships between the best known modal logics, namely logics that can be formed by adding a selection of the axioms (*D*), (*M*), (4), (*B*) and (5) to **K**. A list of these (and other) axioms along with their corresponding frame conditions can be found below the diagram.



In this chart, systems are given by the list of their axioms. So, for example **M4B** is the result of adding (*M*) (4) and (*B*) to **K**. In boldface, we have indicated traditional names of some systems. When system **S** appears below and/or to the left of **S'** connected by a line, then **S'** is an extension of **S**. This means that every argument provable in **S** is provable in **S'**, but **S** is weaker than **S'**, i.e. not all arguments provable in **S'** are provable in **S**.

The following list indicates axioms, their names, and the corresponding conditions on the accessibility relation *R*, for axioms so far discussed in this encyclopedia entry.

Axiom Name	Axiom	Condition on Frames	R is...
(*D*)	$\Box A \to \Diamond A$	$\exists u\ wRu$	Serial
(*M*)	$\Box A \to A$	wRw	Reflexive
(4)	$\Box A \to \Box\Box A$	$(wRv \& vRu) \Rightarrow wRu$	Transitive
(*B*)	$A \to \Box\Diamond A$	$wRv \Rightarrow vRw$	Symmetric
(5)	$\Diamond A \to \Box\Diamond A$	$(wRv \& wRu) \Rightarrow vRu$	Euclidean
(*CD*)	$\Diamond A \to \Box A$	$(wRv \& wRu) \Rightarrow v=u$	Unique
($\Box$*M*)	$\Box(\Box A \to A)$	$wRv \Rightarrow vRv$	Shift Reflexive
(*C4*)	$\Box\Box A \to \Box A$	$wRv \Rightarrow \exists u(wRu \& uRv)$	Dense
(*C*)	$\Diamond\Box A \to \Box\Diamond A$	$wRv \& wRx \Rightarrow \exists u(vRu \& xRu)$	Convergent

Kripke (Relational) Models

$$\mathbb{M} = \langle W, R, V \rangle$$

Kripke (Relational) Models

$$\mathbb{M} = \langle W, R, V \rangle$$

- $W \neq \emptyset$
- $R \subseteq W \times W$
- $V : \mathsf{At} \rightarrow \wp(W)$

Truth in a Kripke Model

1. $\mathbb{M}, w \models p$ iff $w \in V(p)$
2. $\mathbb{M}, w \models \neg\varphi$ iff $\mathbb{M}, w \not\models \varphi$
3. $\mathbb{M}, w \models \varphi \wedge \psi$ iff $\mathbb{M}, w \models \varphi$ and $\mathbb{M}, w \models \psi$
4. $\mathbb{M}, w \models \Box\varphi$ iff for each $v \in W$, if wRv then $\mathbb{M}, v \models \varphi$
5. $\mathbb{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathbb{M}, v \models \varphi$

Some Validities

(M) $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$

(C) $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$

(N) $\Box\top$

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

(Nec) from $\vdash \varphi$ infer $\vdash \Box\varphi$

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Some properties of binary relations

A binary relation $R \subseteq X^2$ is called:

- reflexive if it satisfies $\forall x\, xRx$.
- irreflexive if it satisfies $\forall x\, \neg xRx$.
- serial if it satisfies $\forall x \exists y\, xRy$.
- functional if it satisfies $\forall x \exists ! y\, xRy$,
- symmetric if it satisfies $\forall x \forall y (xRy \rightarrow yRx)$.
- asymmetric if it satisfies $\forall x \forall y (xRy \rightarrow \neg yRx)$.
- antisymmetric if it satisfies $\forall x \forall y (xRy \wedge yRx \rightarrow x = y)$.
- connected if it satisfies $\forall x \forall y (xRy \vee yRx)$.
- transitive if it satisfies $\forall x \forall y \forall z ((xRy \wedge yRz) \rightarrow xRz)$.
- equivalence relation if it is reflexive, symmetric, and transitive.
- euclidean if it satisfies $\forall x \forall y \forall z ((xRy \wedge xRz) \rightarrow yRz)$.
- pre-order, (or quasi-order) if it is reflexive and transitive.
- partial order, if it is reflexive, transitive, and antisymmetric.
- linear order, (or total order) if it is a connected partial order.
- well-founded order, if it is a partial order with no infinite strictly decreasing chains.

Some relational properties of Kripke frames definable by modal formulae

Claim For every Kripke frame $\mathfrak{F} = (W, R)$ the following holds:

- $\mathfrak{F} \models \Box p \rightarrow p$ iff the relation R is reflexive.
  Thus, the formula $\Box p \rightarrow p$ defines the class of reflexive frames.
- $\mathfrak{F} \models \Box p \rightarrow \Diamond p$ iff the relation R is serial.

  Exercise: Find a simpler modal formula that defines seriality.
- $\mathfrak{F} \models \Box p \leftrightarrow \Diamond p$ iff the relation R is a function.
- $\mathfrak{F} \models p \rightarrow \Box\Diamond p$ iff $\mathfrak{F} \models \Diamond\Box p \rightarrow p$ iff
  the relation R is symmetric.
- $\mathfrak{F} \models \Box p \rightarrow \Box\Box p$ iff $\mathfrak{F} \models \Diamond\Diamond p \rightarrow \Diamond p$ iff
  the relation R is transitive.
- $\mathfrak{F} \models \Diamond p \rightarrow \Box\Diamond p$ iff $\mathfrak{F} \models \Diamond\Box p \rightarrow \Box p$ iff
  the relation R is euclidean.

More relational properties of Kripke frames definable by modal formulae

- A challenge: $\mathfrak{F} \models \Diamond\Box p \to \Box\Diamond p$ iff ...?
- A bigger challenge: $\mathfrak{F} \models \Box\Diamond p \to \Diamond\Box p$ iff ...?
- An even bigger challenge: $\mathfrak{F} \models \Box(\Box p \to p) \to \Box p$ iff ...?

Validity of modal formulae

Some valid modal formulae:

- Every modal instance of a propositional tautology, i.e., every formula obtained by uniform substitution of modal formulae for propositional variables in a propositional tautology.

  For instance: $\Box p \vee \neg \Box p$; $(\Box p \wedge \Diamond \Box q) \to \Diamond \Box q$, etc.

- K: $\Box(p \to q) \to (\Box p \to \Box q)$;
- $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$.
- $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$.
- $\Box \varphi$, for every valid modal formula φ.

  E.g., $\Box(\Diamond p \vee \neg \Diamond p)$.

Unlike first-order logic, testing validity in modal logic is decidable, and PSPACE-complete.

Precursors of Intuitionism

- Predecessors of Brouwer: Kronecker: "God made the natural numbers, the rest is human work", French semi-intuitionists (e.g. E. Borel).
- Unhappy feeling from these mathematicians about abstractness of mathematics, proving the existence of objects by reasoning by contradiction, so that no object really arises from the proof: $\neg\forall x \neg Ax \rightarrow \exists x\, Ax$.

Brouwer's ideas

- Foundations unnecessary, in fact impossible,
- Logic follows mathematics, is not its basis, logical rules extracted from mathematics,
- Mathematics is a mental activity, the "exact part of human thought", writing mathematics down is only an aide,
- Criticism of 'classical' logical laws,
- Principle of the excluded third (law of the exclude middle) $A \vee \neg A$.

Example of nonconstructive proof

- Theorem There exist irrational numbers r and s such that r^s is rational.
- Proof Well-known since Euclid, $\sqrt{2}$ is irrational.
- Now either $\sqrt{2}^{\sqrt{2}}$ is rational or it is not.
- In the first case take $r = \sqrt{2}$, $s = \sqrt{2}$. Then $r^s = 2$, i.e. rational.
- In the second, take $r = \sqrt{2}^{\sqrt{2}}$, $s = \sqrt{2}$. Then $r^s = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$, i.e. rational.
- So, we have found r and s as required, only we cannot tell what r is, it is either $\sqrt{2}^{\sqrt{2}}$ or $\sqrt{2}$ (in reality of course the latter) and $= \sqrt{2}$.

Heyting

- Heyting, 1928-1930:
- Earlier incomplete version in Kolmogorov 1925,
- Hilbert type system. We first give natural deduction variant of which first version was given by Gentzen.
- $\neg\varphi$ is defined as $\varphi \rightarrow \bot$ where $\bot$ stands for a contradiction, an obviously false statement like $1 = 0$.

Classical Logic

To get classical logic one adds the rule that if $\bot$ is derived from $\neg\varphi$, then one can conclude to φ dropping the assumption $\neg\varphi$.

$$\begin{array}{c} \varphi \longrightarrow \bot \\ \vdots \\ \dfrac{\bot}{\neg\varphi} \end{array}$$

BHK-interpretation

- Brouwer-Heyting-Kolmogorov Interpretation of connectives and quantifiers.

  Natural deduction closely related to BHK.

- Interpretation by means of proofs (nonformal, nonsyntactical objects, mind constructions),
- A proof of $\varphi \wedge \psi$ consists of proof of φ plus proof of ψ (plus conclusion),
- A proof of $\varphi \vee \psi$ consists of proof of φ or of proof of ψ (plus conclusion),
- A proof of $\varphi \rightarrow \psi$ consists of method that applied to any conceivable proof of φ will deliver proof of ψ,

BHK-interpretation, continued

- Nothing is a proof of $\bot$,
- Proof of $\neg\varphi$ is method that given any proof of φ gives proof of $\bot$,
- A proof of $\exists x\, \varphi(x)$ consists of object d from domain plus proof of $\varphi(d)$ (plus conclusion),
- A proof of $\forall x\, \varphi(x)$ consists of method that applied to any element d of domain will deliver proof of $\varphi(d)$,

Valid and invalid reasoning

- A disjunction is hard to prove: e.g. of the four directions of the de Morgan laws only $\neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi$ is not valid,
- $\neg(\varphi \vee \psi) \rightarrow \neg\varphi \wedge \neg\psi$,
- $(\neg\varphi \wedge \neg\psi) \rightarrow \neg(\varphi \vee \psi)$,
- $\neg\varphi \vee \neg\psi \rightarrow \neg(\varphi \wedge \psi)$ are valid,
- other examples of such invalid formulas are $\varphi \vee \neg\varphi$, (the law of the the excluded middle)
- $\neg(\varphi \wedge \psi) \rightarrow \neg\varphi \vee \neg\psi$,
- $(\varphi \rightarrow \psi \vee \chi) \rightarrow (\varphi \rightarrow \psi) \vee (\varphi \rightarrow \chi)$,
- $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow \varphi \vee \psi$,

Valid and invalid reasoning, continued

- An existential statement is hard to prove:
- of the four directions of the classically valid interactions between negations and quantifiers only $\neg\forall x\,\varphi \rightarrow \exists x \neg\,\varphi$ is not valid,
- statements directly based on the two-valuednes of truth values are not valid, e.g. $\neg\neg\,\varphi \rightarrow \varphi$ or $((\varphi \rightarrow \psi) \rightarrow\ \varphi) \rightarrow \varphi$ (*Peirce's law*),
- On the other hand, many basic laws naturally remain valid, commutativity and associativity of conjunction and disjunction, both distributivity laws,
- $(\varphi \rightarrow \psi \wedge \chi) \leftrightarrow (\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)$,
- $(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \leftrightarrow (\varphi \vee \psi \rightarrow \chi))$,
- $(\varphi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow (\varphi \wedge \psi) \rightarrow \chi$,
- $((\varphi \vee \psi) \wedge \neg\varphi \rightarrow \psi))$ (needs *ex falso*!).

Hilbert type system

- $\varphi \to (\psi \to \varphi)$
- $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)))$
- The only rule is modus ponens from φ and $\varphi \to \psi$ conclude ψ.
- The first two axioms plus modus ponens are sufficient for proving the deduction theorem. (corresponding to implication introduction).
- $\varphi \wedge \psi \to \varphi \quad \varphi \wedge \psi \to \psi$,
- $\varphi \to (\psi \to \varphi \wedge \psi)$,
- $\varphi \to \varphi \vee \psi \quad \psi \to \varphi \vee \psi$,
- $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$,
- $\bot \to \varphi$,

Classical propositional calculus

- To get CPC add $((\varphi \to \psi) \to \varphi) \to \varphi$ (Peirce's law) or $\neg\neg\varphi \to \varphi$.

Kripke frames and models

- Frames, (usually $\mathfrak{F}$):
- A set of worlds W, also nodes, points
- An accessibility relation R, which is a $\leqslant$-partial order,
- For models $\mathfrak{M}$ a persistent valuation V is added. Persistence means:
- $wRw' \,\&\, w \in V(p) \Longrightarrow w \in V'(p)$.
- $w \models \varphi \wedge \psi \Longleftrightarrow w \models \varphi$ and $w \models \psi$,
- $w \models \varphi \vee \psi \Longleftrightarrow w \models \varphi$ or $w \models \psi$,
- $w \models \varphi \to \psi \Longleftrightarrow \forall w'(wRw'$ and $w' \models \varphi \Rightarrow w' \models \psi)$,

Kripke frames and models, continued

- Frames will usually have a root w_0: $w_0\,R\,w$ for all w.
- $w \not\models \bot$,
- $w \models \neg\varphi \iff \forall w'(wRw' \Rightarrow$ not $w \models \varphi)$ (follows from definition of $\neg\varphi$ as $\varphi \rightarrow \bot$),
- Persistence for formulas follows:
- $wRw' \,\&\, w \models \varphi \Longrightarrow w' \models \varphi$.
- Note that $w \models \neg\neg\varphi \iff \forall w'(wRw' \Longrightarrow \exists w''(w'Rw'' \,\&\, w'' \models \varphi))$
- $\Leftrightarrow$ for finite models $\leftrightarrow \forall w''(wRw'' \,\&\, w''$ end point $\Longrightarrow w'' \models \varphi)$.

Kripke frames and models, predicate logic

- Increasing domains D_w:
- $wRw' \Longrightarrow D_w \subseteq D_{w'}$.
- with names for the elements of the domains:
- $w \models \exists x\varphi(x) \iff$, for some $d \in D_w$, $w \models \varphi(d)$,
- $w \models \forall x\varphi(x) \iff$, for each w' with wRw' and all $d \in D_{w'}$, $w' \models \varphi(d)$,
- Persistency transfers to formulas here as well.

Counter-models to propositional formulas

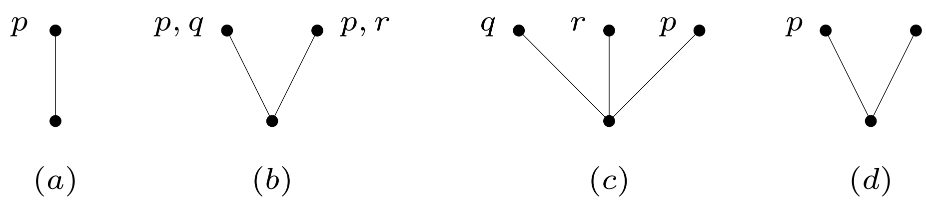


Figure 1: Counter-models for the propositional formulas

- These figures give counterexamples to respectively:
- (a) $p \vee \neg p$, $\neg\neg p \to p$,
- (b) $(p \to q \vee r) \to (p \to q) \vee (p \to r)$,
- (c) $(\neg p \to q \vee r) \to (\neg p \to q) \vee (p \to r)$,
- (d) $(\neg\neg p \to p) \to p \vee \neg p$.

Counter-models to predicate formulas

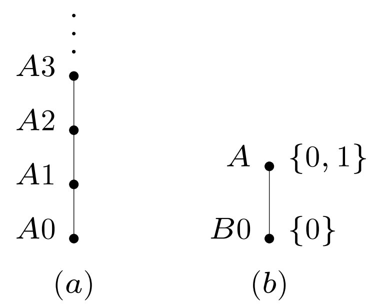


Figure 2: Counter-models for the predicate formulas

- These figures give counterexamples to:
- (a) $\neg\neg\forall x(Ax \vee \neg Ax)$, if domain constant $\mathbb{N}$ (and also against $\forall x\neg\neg Ax \rightarrow \neg\neg\forall x Ax$),
- (b) $\forall x(A \vee Bx) \rightarrow A \vee \forall x Bx$.

Glivenko's theorem

- Before the completeness proof an application of completeness.
- Glivenko's Theorem, Theorem 5:
- $\vdash_{\mathrm{CPC}} \varphi$ iff $\vdash_{\mathrm{IPC}} \neg\neg\varphi$ (CPC is classical propositional calculus).
- $\Longleftarrow$ is of course trivial.
- $\Longrightarrow$ Exercise.
- e.g. $\vdash_{\mathrm{IPC}} \neg\neg(\varphi \vee \neg\varphi)$.
- Glivenko's Theorem does not extend to predicate logic, exercise.

Disjunction property

- Theorem 16. $\vdash_{\mathrm{IPC}} \varphi \vee \psi$ iff $\vdash_{\mathrm{IPC}} \varphi$ or $\vdash_{\mathrm{IPC}} \psi$.
- This extends to the predicate calculus and arithmetic.
- Proof. $\Leftarrow$: Trivial

  $\Rightarrow$: Assume $\nvdash_{\mathrm{IPC}} \varphi$ and $\nvdash_{\mathrm{IPC}} \psi$.
- Let $\mathfrak{K} \not\models \varphi$ and $\mathfrak{L} \not\models \psi$.
- Add a new root w_0 below both $\mathfrak{K}$ and $\mathfrak{L}$. In w_0, $\varphi \vee \psi$ is falsified (because of persistence!).

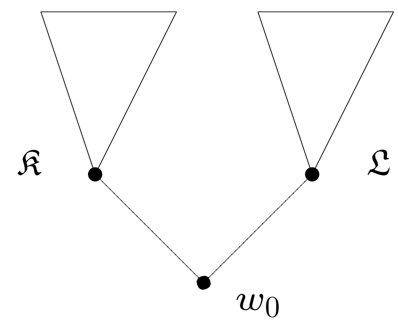


Figure 3: Proving the disjunction property

Kripke frames, models for S4, Grz and GL

- S4 characterizes the reflexive transitive frames,
- S4 is complete w.r.t. the (finite) reflexive, transitive frames,
- S4 is complete w.r.t. $\leqslant$-partial orders (reflexive, transitive, anti-symmetric)
- Grz characterizes the reflexive, transitive, conversely well-founded frames,
- Grz is complete w.r.t. the finite $\leqslant$-partial orders,
- GL characterizes the transitive, conversely well-founded (i.e. irreflexive, asymmetric) frames.
- GL is complete w.r.t. the finite $<$-partial orders.

Translations

- Gödel's negative translation
- extends to the predicate calculus and arithmetic, has many variations,
- Definition 28
- $p^n = \neg\neg p$,
- $(\varphi \wedge \psi)^n = \varphi^n \wedge \psi^n$,
- $(\varphi \vee \psi)^n = \neg\neg(\varphi^n \vee \psi^n)$,
- $(\varphi \rightarrow \psi)^n = \varphi^n \rightarrow \psi^n$,
- $\bot^n = \bot$.

Properties of Gödel's negative translation

- Theorem 29. $\vdash_{\mathrm{CPC}} \varphi$ iff $\vdash_{\mathrm{IPC}} \varphi^n$.
- Proof.
- $\Longleftarrow$: $\vdash_{\mathrm{IPC}} \varphi^n \Rightarrow \vdash_{\mathrm{CPC}} \varphi^n \Rightarrow \vdash_{\mathrm{CPC}} \varphi$.

  $\Longrightarrow$: First prove $\vdash_{\mathrm{IPC}} \varphi^n \leftrightarrow \neg\neg\varphi^n$ (φ^n is negative) (using $\vdash_{\mathrm{IPC}} \neg\neg(\varphi \to \psi) \leftrightarrow (\neg\neg\varphi \to \neg\neg\psi)$ and $\vdash_{\mathrm{IPC}} \neg\neg(\varphi \wedge \psi) \leftrightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$. Then simply follow the proof of φ in CPC to mimic it with a proof of φ^n in IPC. Exercise.

Gödel's translation of IPC into S4

- Gödel noticed the closeness of S4 and IPC when one interprets $\Box$ as intuitive provability.
- Definition 32.
- $p^{\Box} = \Box p$,
- $(\varphi \wedge \psi)^{\Box} = \varphi^{\Box} \wedge \psi^{\Box}$,
- $(\varphi \vee \psi)^{\Box} = \varphi^{\Box} \vee \psi^{\Box}$,
- $(\varphi \rightarrow \psi)^{\Box} = \Box(\varphi^{\Box} \rightarrow \psi^{\Box})$,
- Theorem 33 $\vdash_{\mathrm{IPC}} \varphi$ iff $\vdash_{\mathrm{S4}} \varphi^{\Box}$ iff $\vdash_{\mathrm{Grz}} \varphi^{\Box}$.

Proof for Gödel's translation of IPC into S4

- Proof $\Longrightarrow$: Trivial from S4 to Grz. From IPC to S4 it is simply a matter of using one of the proof systems of IPC and to find the needed proofs in S4, or showing their validity in the S4-frames and using completeness.

- $\Longleftarrow$: It is sufficient to note that it is easily provable by induction on the length of the formula φ that for any world w in a Kripke model with a persistent valuation $w \models \varphi$ iff $w \models \varphi^{\Box}$. This means that if $\nvdash_{\mathrm{IPC}} \varphi$ one can interpret the finite IPC-countermodel to φ provided by the completeness theorem immediately as a finite Grz-countermodel to $\varphi^{\Box}$.

Intermediate Logics

- Intermediate logics (Superintuitionistic logics),
- Logics extending intuitionistic logic by axiom schemes (and sublogics of classical logic),
- e.g. Weak excluded middle: $\neg\varphi \vee \neg\neg\varphi$,
- Dummett's logic: $(\varphi \to \psi) \vee (\psi \to \varphi)$,
- most do not have disjunction property, some do:
- e.g. the Kreisel-Putnam logic $(\neg\varphi \to \psi \vee \chi) \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$,

Modal Logics Weaker than **K**

A semantics for modal logics:

Lindenbaum-Tarski (Boolean) Algebras

$\mathcal{B} = (B, \wedge, \vee, ', \leqslant, 0, 1, \boxdot)$ $\quad \boxdot : B \to B$

Let T be a theory. $[\varphi]_T = \{\psi \mid T \vdash \varphi \leftrightarrow \psi\}$.

$[\varphi]_T \wedge [\psi]_T = [\varphi \wedge \psi]_T$ $\quad$ $[\varphi]_T \vee [\psi]_T = [\varphi \vee \psi]_T$

$[\varphi]'_T = [\neg\varphi]_T$ $\quad$ $[\varphi]_T \leqslant [\psi]_T$ iff $T \vdash \varphi \to \psi$;

$0 = [\bot]_T \qquad 1 = [\top]_T$ $\quad$ $\boxdot[\varphi]_T = [\Box\varphi]_T$.

$$\text{Well--defined iff} \quad \frac{T \vdash \varphi \leftrightarrow \psi}{T \vdash \Box\varphi \leftrightarrow \Box\psi}.$$

Minimal Modal Logic **E**

CPC + Rule of Inference

$$(\mathtt{RE})\ \frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}.$$

Monotone Modal Logic **M**

CPC + Monotonicity Rule

$$(\mathtt{RM})\ \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$$

(or equivalently) **E** + the Axiom

$$(\mathtt{M})\ \Box(A \wedge B) \to \Box A \wedge \Box B.$$

Necessitation Modal Logic **N**

CPC + Necessitation Rule

$$(\mathtt{RN})\ \frac{\varphi}{\Box\varphi}$$

(or equivalently) **E** + the Axiom

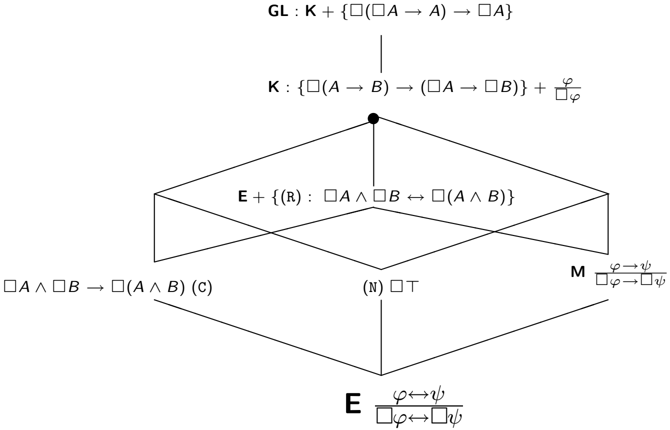
$$(\mathtt{N})\ \Box\top.$$

Axiom $(\mathtt{C})\ \Box A \wedge \Box B \to \Box(A \wedge B)$ converse of monotonicity

---

$$\mathbf{K} = \mathbf{E} + (\mathtt{N}) + (\mathtt{M}) + (\mathtt{C}) = \mathbf{M} + \mathbf{N} + \mathtt{C}$$

Non-Normal Modal Logics



Literature:
B. Chellas, *Modal Logic: An Introduction*, CUP 1990.

Philosophically ...?

No (explicit) mention in the **Handbook of Modal Logic**?

Proof–Theoretic Aspects [e.g. cut elimination] Different Systems

Let $\Box\varphi$ mean

- happening of φ with high probability
- having a strategy to force φ
- the set of consequences of φ
- cut-free provability of φ in weak arithmetics

then $\Box$ does not satisfy $(\mathtt{K})$.

High Probability

Fix a threshold $r < 1$ and let $\Box\varphi$ mean

happening of φ with probability $\geq r$.

Take an $1 \leq x < 1/\sqrt{r}$, and assume ϕ and ψ are independent with probability $x \cdot r$. Then $\Box\phi \wedge \Box\psi$.
But $\Box(\phi \wedge \psi)$ does not hold, because the probability of $\phi \wedge \psi$ is $x^2 \cdot r^2 < (1/r) \cdot r^2 = r$.
Thus $(\mathtt{C}) : \Box\phi \wedge \Box\psi \not\to \Box(\phi \wedge \psi)$ under this interpretation.

Though $(\mathtt{RE})$: $A \leftrightarrow B / \Box A \leftrightarrow \Box B$, $(\mathtt{M})$: $\Box(A \wedge B) \to \Box A \wedge \Box B$, and $(\mathtt{N})$: $\Box\top$ are valid.

Deductive Closure

For Σ a set of sentences in CPC, a Σ-valuation is a mapping $*$
$(A \wedge B)^* = A^* \cap B^*$, $(\neg A)^* = \Sigma - A^*$, and
$(\Box A)^* = \{\alpha \in \Sigma \mid A^* \vdash_{CPC} \alpha\}$.

This modal logic can be axiomatized by

- ▷ $A \rightarrow \Box A$ — reflexivity
- ▷ $\Box(A \vee \Box A) \rightarrow \Box A$ — transitivity
- ▷ $A \rightarrow B / \Box A \rightarrow \Box B$ — monotonicity

because

- ◁ $A^* \subseteq (\Box A)^*$
- ◁ $(\Box(A \vee \Box A))^* \subseteq (\Box A)^*$
- ◁ if $A^* \subseteq B^*$ then $(\Box A)^* \subseteq (\Box B)^*$

Deductive Closure

Proof of Completeness in
[P. Naumov, "On modal logic of deductive closure", *APAL* (2006)]

For $(\mathtt{C}) : \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$ we should have
$(\Box A)^* \cap (\Box B)^* \subseteq (\Box(A \wedge B))^*$ which is not true:
$A^* \vdash \alpha \ \ \& \ \ B^* \vdash \alpha \not\longrightarrow \ \ A^* \cap B^* \vdash \alpha$
(put $A^* = \{\mathfrak{p}\}$, $B^* = \{\mathfrak{q}\}$, and $\alpha = \mathfrak{p} \vee \mathfrak{q}$).
Thus $\Box A \wedge \Box B \not\rightarrow \Box(A \wedge B)$.

Also $(\mathtt{N}) : \Box\top$, because $\{\alpha \in \Sigma \mid \Sigma \vdash \alpha\} = \Sigma$.

Neighborhoods in Topology

In a topology, a *neighborhood* of a point x is any set A containing x such that you can "wiggle" x without leaving A.

A *neighborhood system* of a point x is the collection of neighborhoods of x.

J. Dugundji. *Topology*. 1966.

Neighborhoods in Modal Logic

Neighborhood Structure: $\langle W, N, V\rangle$

- $W \neq \emptyset$
- $N : W \to \wp(\wp(W))$
- $V : \mathsf{At} \to \wp(W)$

Some Notation

Given $\varphi \in \mathcal{L}$ and a model $\mathbb{M}$, the

- *proposition* expressed by φ
- *extension* of φ
- *truth set* of φ

is

Some Notation

Given $\varphi \in \mathcal{L}$ and a model $\mathbb{M}$, the

- *proposition* expressed by φ
- *extension* of φ
- *truth set* of φ

is

$$(\varphi)^{\mathbb{M}} = \{w \in W \mid \mathbb{M}, w \models \varphi\}$$

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

What does it mean to be a neighborhood?

Neighborhood Models: $\mathcal{M} = (W, \mathrm{N}, \mathscr{V})$
where $\mathrm{N} : W \to \mathscr{P}\mathscr{P}(W)$ - neighborhood function; and

$\mathscr{V} : \mathrm{Atomic} \to \mathscr{P}(W)$ which can be extended to all formulae:

$\mathscr{V}(\neg\phi) = W - \mathscr{V}(\phi)$; $\mathscr{V}(\phi \wedge \psi) = \mathscr{V}(\phi) \cap \mathscr{V}(\psi)$; and

$\mathscr{V}(\Box\phi) = \{w \in W \mid \mathscr{V}(\phi) \in \mathrm{N}(w)\}$.

I.O.W. $w \models \Box\phi \quad \Leftrightarrow \quad \{v \in W \mid v \models \phi\} \in \mathrm{N}(\phi)$.

Then RE: $A \leftrightarrow B / \Box A \leftrightarrow \Box B$ is valid in every Neighborhood model.
The Logic of Neighborhood Models is **E** ($\subseteq$ Classical).

M $\langle \overline{\underline{\textit{sound\&complete}}} \rangle$ each $\mathrm{N}(w)$ closed under superset

---

N $\langle \overline{\underline{\textit{sound\&complete}}} \rangle$ each $\mathrm{N}(w) \ni W$

C $\langle \overline{\underline{\textit{sound\&complete}}} \rangle$ each $\mathrm{N}(w)$ closed under intersection

---

K $\langle \overline{\underline{\textit{sound\&complete}}} \rangle$ each $\mathrm{N}(w)$ is a filter

Neighborhood Models

There is more ...

For a Kripke Model $(W, R, \vDash)$ let $(W, \aleph, \mathscr{V})$ be defined:
$\aleph(w) = \Big\{X \subseteq W \mid X \supseteq \{v \in W \mid wRv\}\Big\}$ and
$\mathscr{V}(\phi) = \{w \in W \mid w \vDash \phi\}$.

Then each $\aleph(w)$ is a [principal] filter.

Eric Pacuit:
Neighborhood Semantics for Modal Logic
An Introduction
Course at ESSLLI 2007

Neighborhood Frames

Let W be a non-empty set of states.

Any map $N : W \to \wp\wp W$ is called a neighborhood function

Definition

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

Neighborhood Model

Let $\mathfrak{F} = \langle W, N \rangle$ be a neighborhood frame. A neighborhood model based on $\mathfrak{F}$ is a tuple $\langle W, N, V \rangle$ where $V : \mathsf{At} \rightarrow 2^W$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\varphi$ iff $(\varphi)^{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - (\varphi)^{\mathfrak{M}} \notin N(w)$

where $(\varphi)^{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $(p)^{\mathfrak{M}} = V(p)$ for $p \in \mathsf{At}$
2. $(\neg\varphi)^{\mathfrak{M}} = W - (\varphi)^{\mathfrak{M}}$
3. $(\varphi \wedge \psi)^{\mathfrak{M}} = (\varphi)^{\mathfrak{M}} \cap (\psi)^{\mathfrak{M}}$
4. $(\Box\varphi)^{\mathfrak{M}} = m_N((\varphi)^{\mathfrak{M}})$
5. $(\Diamond\varphi)^{\mathfrak{M}} = W - m_N(W - (\varphi)^{\mathfrak{M}})$

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

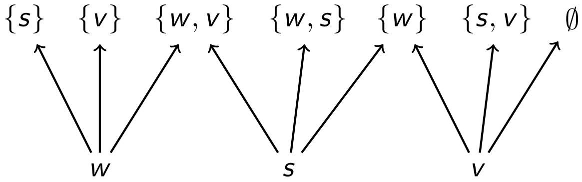
Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

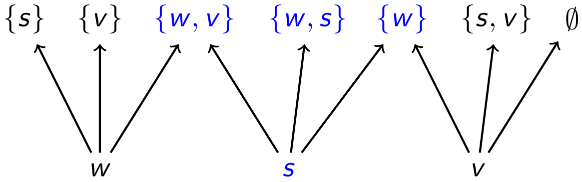


Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

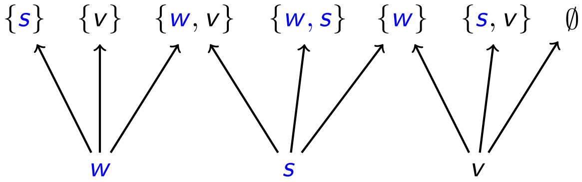


Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

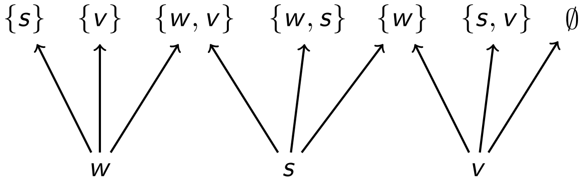
- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.



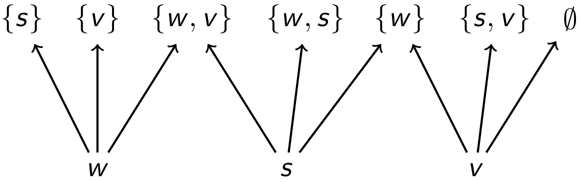
Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



Detailed Example

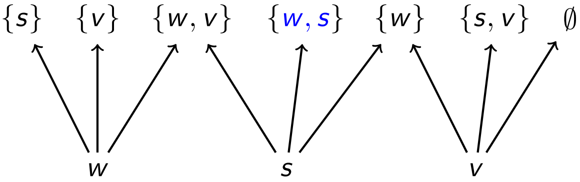
$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$$\mathfrak{M}, s \models \Box p$$

Detailed Example

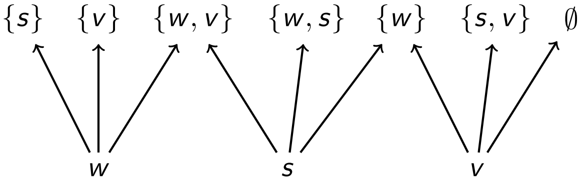
$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$$\mathfrak{M}, s \models \Box p$$

Detailed Example

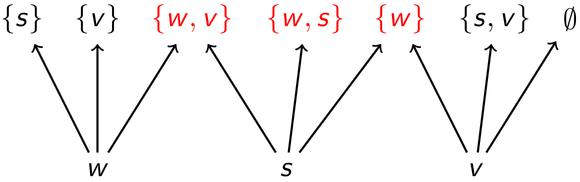
$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$$\mathfrak{M}, s \models \Diamond p$$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$

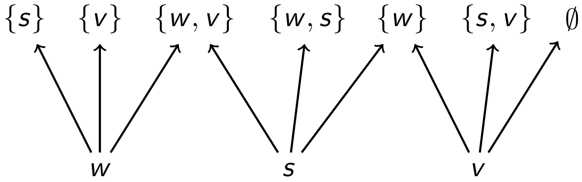


$$\mathfrak{M}, s \models \Diamond p$$

$$(\neg p)^{\mathfrak{M}} = \{v\}$$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \models \Diamond\Box p$?

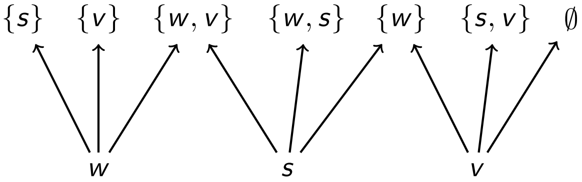
$\mathfrak{M}, w \models \Box\Box p$?

$\mathfrak{M}, v \models \Box\Diamond p$?

$\mathfrak{M}, v \models \Diamond\Box p$?

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \models \Diamond\Box p$?

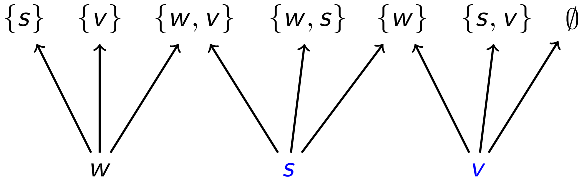
$\mathfrak{M}, w \models \Box\Box p$?

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$?

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \models \Diamond\Box p$?

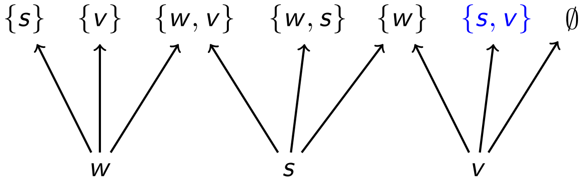
$\mathfrak{M}, w \models \Box\Box p$?

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$?

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \models \Diamond\Box p$?

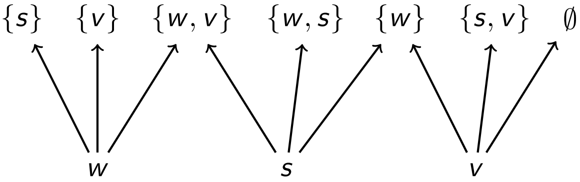
$\mathfrak{M}, w \models \Box\Box p$?

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$?

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \not\models \Diamond\Box p$

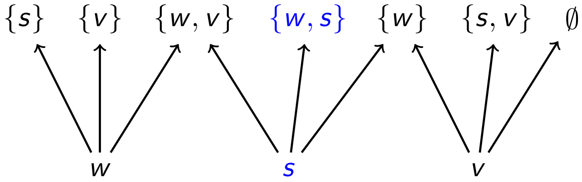
$\mathfrak{M}, w \models \Box\Box p$

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \not\models \Diamond\Box p$

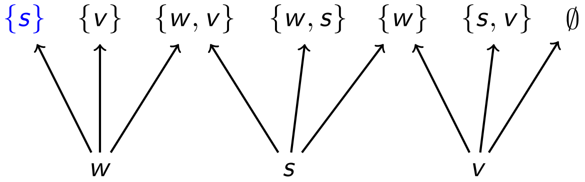
$\mathfrak{M}, w \models \Box\Box p$

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \not\models \Diamond\Box p$

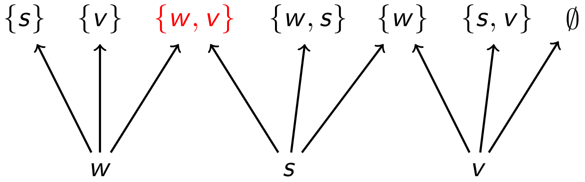
$\mathfrak{M}, w \models \Box\Box p$

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



$\mathfrak{M}, w \not\models \Diamond\Box p$

$\mathfrak{M}, w \models \Box\Box p$

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$

New slogan: The basic modal language is a simple language for talking about *neighborhood structures*.

What can we say?

Definition

A modal formula φ defines a property P of neighborhood functions if any neighborhood frame $\mathfrak{F}$ has property P iff $\mathfrak{F}$ validates φ.

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ *iff* $\mathfrak{F}$ *is closed under supersets.*

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ *iff* $\mathfrak{F}$ *is closed under supersets.*

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ *iff* $\mathfrak{F}$ *is closed under finite intersections.*

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

On relational frames, these formulas both define the same property: seriality.

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

On relational frames, these formulas both define the same property: seriality.

On neighborhood frames:

- $\Diamond\top$ corresponds to the property $\emptyset \notin N(w)$

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

On relational frames, these formulas both define the same property: seriality.

On neighborhood frames:

- $\Diamond\top$ corresponds to the property $\emptyset \notin N(w)$
- $\Box\varphi \rightarrow \Diamond\varphi$ is valid on $\mathfrak{F}$ iff $\mathfrak{F}$ is proper.

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ be a neighborhood frame such that for each $w \in W$, $N(w) \neq \emptyset$.

1. $\mathfrak{F} \models \Box\varphi \to \varphi$ *iff for each* $w \in W$, $w \in \cap N(w)$
2. $\mathfrak{F} \models \Box\varphi \to \Box\Box\varphi$ *iff for each* $w \in W$, *if* $X \in N(w)$, *then* $\{v \mid X \in N(v)\} \in N(w)$

Find properties on frames that are defined by the following formulas:

1. $\Box\bot$
2. $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$
3. $\Diamond\varphi \rightarrow \Box\varphi$
4. $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$
5. $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

Some Non-validities

1. $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
2. $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
3. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
4. $\Box\top$
5. $\Box\varphi \rightarrow \varphi$
6. $\Box\varphi \rightarrow \Box\Box\varphi$
7. Many more...

Validities

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ is valid in all neighborhood models.

(Re) If $\varphi \leftrightarrow \psi$ is valid then $\Box\varphi \leftrightarrow \Box\psi$ is valid.

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

E is the smallest classical modal logic.

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

In **E**, *M* is equivalent to

$$(Mon)\ \frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

PC 6. Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

A logic is normal if it contains all instances of *E*, *C* and is closed under *Mon* and *Nec*

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K is the smallest normal modal logic

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K = **EMCN**

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K $= PC(+E) + K + Nec + MP$

Are there non-normal extensions of **K**?

Are there non-normal extensions of **K**? Yes!

Are there non-normal extensions of **K**? Yes!

Let **L** be the smallest modal logic containing

- **S4** (**K** $+ \;\Box\varphi \to \varphi \;+\; \Box\varphi \to \Box\Box\varphi$)
- all instances of M: $\Box\Diamond\varphi \to \Diamond\Box\varphi$

Claim: **L** is a non-normal extension of **S4**.

Useful Fact

Theorem (Uniform Substitution)

The following rule can be derived in **E**

$$\frac{\psi \leftrightarrow \psi'}{\varphi \leftrightarrow \varphi[\psi/\psi']}$$

Interesting Fact

Each of K, M and C are logically independent:

- $\mathbf{EC} \not\vdash K$
- $\mathbf{EM} \not\vdash K$
- $\mathbf{EK} \not\vdash M$
- $\mathbf{EK} \not\vdash C$

Interesting Fact

Each of K, M and C are logically independent:

- $\mathbf{EC} \nvdash K$
- $\mathbf{EM} \nvdash K$
- $\mathbf{EK} \nvdash M$
- $\mathbf{EK} \nvdash C$

"Our discussion indicates that, in a sense, C is a more fundamental schema than K; yet it is K which is most often used in axiomatizations of normal modal logics."

(pg. 45)

K. Segerberg. *An Essay on Classical Modal Logic*. 1970.

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics?

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics? Yes!

Comparing Relational and Neighborhood Semantics

There is

- an extension of **K**

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

Comparing Relational and Neighborhood Semantics

There is

- an extension of **K**

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

- An extension of **T**

M. Gerson. *A Neighbourhood frame for T with no equivalent relational frame*. Zeitschr. J. Math. Logik und Grundlagen (1976).

Comparing Relational and Neighborhood Semantics

There is

- an extension of **K**

  D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

- An extension of **T**

  M. Gerson. *A Neighbourhood frame for T with no equivalent relational frame*. Zeitschr. J. Math. Logik und Grundlagen (1976).

- An extension of **S4**

  M. Gerson. *An Extension of S4 Complete for the Neighbourhood Semantics but Incomplete for the Relational Semantics*. Studia Logica (1975).

The general situation is not very well understood.

The general situation is not very well understood.

Notable exceptions:

L. Chagrova. *On the Degree of Neighborhood Incompleteness of Normal Modal Logics*. AiML 1 (1998).

V. Shehtman. *On Strong Neighbourhood Completeness of Modal and Intermediate Propositional Logics (Part I)*. AiML 1 (1998).

T. Litak. *Modal Incompleteness Revisited*. Studia Logica (2004).

Topological Models for Modal Logic

Definition

Topological Space A **topological space** is a neighborhood frame $\langle W, \mathcal{T} \rangle$ where W is a nonempty set and

1. $W \in \mathcal{T}$, $\emptyset \in W$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

Topological Models for Modal Logic

Definition

Topological Space A **topological space** is a neighborhood frame $\langle W, \mathcal{T} \rangle$ where W is a nonempty set and

1. $W \in \mathcal{T}$, $\emptyset \in W$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

A neighborhood of w is any set X such that there is an $O \in \mathcal{T}$ with $w \in O \subseteq N$

Let $\mathcal{T}_w$ be the collection of all neighborhoods of w.

Topological Models for Modal Logic

Definition

Topological Space A **topological space** is a neighborhood frame $\langle W, \mathcal{T} \rangle$ where W is a nonempty set and

1. $W \in \mathcal{T}$, $\emptyset \in W$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

Lemma

Let $\langle W, \mathcal{T} \rangle$ be a topological space. Then for each $w \in W$, the collection $\mathcal{T}_w$ contains W, is closed under finite intersections and closed under arbitrary unions.

Topological Models for Modal Logic

The largest open subset of X is called the interior of X, denoted $Int(X)$. Formally,

$$Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$$

The smallest closed set containing X is called the closure of X, denoted $Cl(X)$. Formally,

$$Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$$

Topological Models for Modal Logic

- $Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$
- $Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$

Lemma

Let $\langle W, \mathcal{T}\rangle$ be a topological space and $X \subseteq W$. Then

1. $Int(X \cap Y) = Int(X) \cap Int(Y)$
2. $Int(\emptyset) = \emptyset$, $Int(W) = W$
3. $Int(X) \subseteq X$
4. $Int(Int(X)) = Int(X)$
5. $Int(X) = W - Cl(W - X)$

Topological Models for Modal Logic

- $Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$
- $Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$

Lemma

Let $\langle W, \mathcal{T}\rangle$ be a topological space and $X \subseteq W$. Then

1. $\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi$
2. $\Box\bot \leftrightarrow \bot, \Box\top \leftrightarrow \top$
3. $\Box\varphi \rightarrow \varphi$
4. $\Box\Box\varphi \leftrightarrow \Box\varphi$
5. $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Topological Models for Modal Logic

A topological model is a triple $\langle W, \mathcal{T}, V\rangle$ where $\langle W, \mathcal{T}\rangle$ is a topological space and V a valuation function.

Topological Models for Modal Logic

A topological model is a triple $\langle W, \mathcal{T}, V\rangle$ where $\langle W, \mathcal{T}\rangle$ is a topological space and V a valuation function.

$$\mathbb{M}^T, w \models \Box\varphi \text{ iff } \exists O \in \mathcal{T}, w \in O \text{ such that } \forall v \in O, \mathbb{M}^T, v \models \varphi$$

$$(\Box\varphi)^{\mathbb{M}^T} = Int((\varphi)^{\mathbb{M}^T})$$

From Neighborhoods to Topologies

From Neighborhoods to Topologies

A family $\mathcal{B}$ of subsets of W is called a basis for a topology $\mathcal{T}$ if every open set can be represented as the union of elements of a subset of $\mathcal{B}$

From Neighborhoods to Topologies

A family $\mathcal{B}$ of subsets of W is called a basis for a topology $\mathcal{T}$ if every open set can be represented as the union of elements of a subset of $\mathcal{B}$

Fact: A family $\mathcal{B}$ of subsets of W is a basis for some topology if

- for each $w \in W$ there is a $U \in \mathcal{B}$ such that $w \in U$
- for each $U, V \in \mathcal{B}$, if $w \in U \cap V$ then there is a $W \in \mathcal{B}$ such that $w \in W \subseteq U \cap V$

From Neighborhoods to Topologies

A family $\mathcal{B}$ of subsets of W is called a basis for a topology $\mathcal{T}$ if every open set can be represented as the union of elements of a subset of $\mathcal{B}$

Let $\mathbb{M} = \langle W, N, V \rangle$ be a neighborhood models. Suppose that N satisfies the following properties

- for each $w \in W$, $N(w)$ is a filter
- for each $w \in W$, $w \in \cap N(w)$
- for each $w \in W$ and $X \subseteq W$, if $X \in N(w)$, then $m_N(X) \in N(w)$

Then there is a topological model that is point-wise equivalent to $\mathbb{M}$.

Main Completeness Result

Theorem

S4 *is the logic of the class of all topological spaces.*

J. van Benthem and G. Bezhanishvili. *Modal Logics of Space*. Handbook of Spatial Logics (2007).

Relations to Kripke Models

Given a Kripke model $\mathcal{K} = (W, R, \Vdash)$ define $\mathcal{M} = \langle W, N, \| \cdot \| \rangle$ by $\|p\| = \{w \in W \mid w \Vdash p\}$, and $N_w = \big\{ X \subseteq W \mid X \supseteq \{v \in W \mid wRv\} \big\}$ (principal) filter.

For any modal formula A, $w \in \|A\| \iff w \Vdash A$.

If in $\mathcal{M} = \langle W, N, \| \cdot \| \rangle$ each N_w is a principal filter, define Kripke model $\mathcal{K} = (W, R, \Vdash)$ by $wRv \iff v \in \bigcap N_w$, and $w \Vdash p \iff w \in \|p\|$.

For any modal formula A, $w \Vdash A \iff w \in \|A\|$.

Modal Logics (Normal & Non-Normal) Modal Logic and Computer Science, IPM, Tehran, 2010

Thank You!

Thanks to the Participants
and
The Organizers of the

IPM Workshop on
Modal Logic and Computer Science

SAEEDSALEHI.ir