A Quick Introduction to MATHEMATICAL LOGIC

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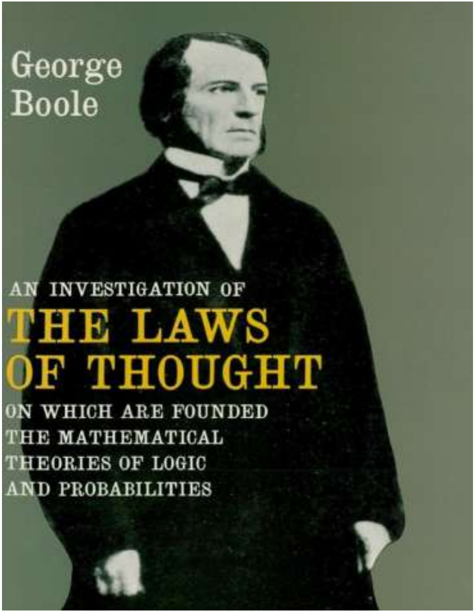
Boolean Algebras, 23 August 2021



AN INVESTIGATION

OF

THE LAWS OF THOUGHT

ON WHICH ARE FOUNDED

THE MATHEMATICAL THEORIES OF LOGIC

AND PROBABILITIES

BY

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Boolean Algebras

Associativity

$$a \wedge (b \wedge c) \equiv (a \wedge b) \wedge c, \quad a \vee (b \vee c) \equiv (a \vee b) \vee c$$

Commutativity

$$a \wedge b \equiv b \wedge a, \quad a \vee b \equiv b \vee a$$

Distributivity

$$a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c), \quad a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$$

Idempotence

$$a \wedge a \equiv a, \quad a \vee a \equiv a$$

Truth and Falsum

$$a \vee (\neg a) \equiv \top, \quad a \wedge \top \equiv a, \quad a \wedge (\neg a) \equiv \bot, \quad a \vee \bot \equiv a$$

de Morgan's Laws

$$\neg(a \wedge b) \equiv (\neg a) \vee (\neg b), \quad \neg(a \vee b) \equiv (\neg a) \wedge (\neg b)$$

More on Boolean Algebras

Example

(i) It immediately follows from the axioms that
$a \equiv a \wedge \top \equiv a \wedge (p \vee \neg p) \equiv (a \wedge p) \vee (a \wedge \neg p)$.

(ii) The *absorbing properties* of truth and falsum, i.e., $a \vee \top \equiv \top$ and $a \wedge \bot \equiv \bot$ follow also from the axioms. We show the former:
$a \vee \top \equiv a \vee (a \vee \neg a) \equiv (a \vee a) \vee (\neg a) \equiv a \vee (\neg a) \equiv \top$.

(iii) One can also prove *the absorption laws*: $a \wedge (a \vee b) \equiv a$ and $a \vee (a \wedge b) \equiv a$. Let us show the latter by using (ii) above:
$a \vee (a \wedge b) \equiv (a \wedge \top) \vee (a \wedge b) \equiv a \wedge (\top \vee b) \equiv a \wedge (b \vee \top) \equiv a \wedge \top \equiv a$.

(iv) The *double negation law* $\neg\neg a \equiv a$ can be proved as follows:
$\neg\neg a \equiv (\neg\neg a) \wedge \top \equiv (\neg\neg a) \wedge (a \vee \neg a) \equiv (\neg\neg a \wedge a) \vee (\neg\neg a \wedge \neg a) \equiv$
$(\neg\neg a \wedge a) \vee (\bot) \equiv (a \wedge \neg\neg a) \vee (a \wedge \neg a) \equiv a \wedge (\neg\neg a \vee \neg a) \equiv a \wedge \top \equiv a$.

Propositional Logic by Boolean Algebras

(Classical) Logic is . . .

$$\boxed{P \rightarrow Q \quad \equiv \quad \neg P \vee Q}$$

$$(p \rightarrow p \vee q) \equiv (\neg p \vee [p \vee q]) \equiv ([p \vee \neg p] \vee q) \equiv (\top \vee q) \equiv \top$$

$$(p \wedge q \rightarrow p) \equiv (\neg [p \wedge q] \vee p) \equiv ([\neg p \vee \neg q] \vee p) \equiv (\top \vee \neg q) \equiv \top$$

$$\boxed{P \leftrightarrow Q \quad \equiv \quad (\neg P \vee Q) \wedge (P \vee \neg Q)}$$

$$\neg(p \leftrightarrow q) \equiv \neg[(\neg p \vee q) \wedge (p \vee \neg q)] \equiv \neg(\neg p \vee q) \vee \neg(p \vee \neg q) \equiv$$
$$(p \wedge \neg q) \vee (\neg p \wedge q) \equiv (p \vee q) \wedge (\neg p \vee \neg q) \equiv (\neg p \leftrightarrow q) \equiv (p \leftrightarrow \neg q)$$

A Puzzle by Boolean Algebras

P says that "*Q is lying*", and Q says that "*both P and Q tell the truth*".
Who is lying and who tells the truth?

► P says $\neg Q$ — Q says $P \wedge Q$ ◄

$P \equiv \neg Q$ — $Q \equiv P \wedge Q$

$$\begin{cases} P \equiv \neg Q \equiv \neg(P \wedge Q) \equiv \neg P \vee \neg Q \equiv \neg\neg Q \vee \neg Q \equiv \top. \\ Q \equiv P \wedge Q \equiv \neg Q \wedge Q \equiv \bot. \end{cases}$$

(P says $\neg Q$) and (Q says $P \wedge Q$) imply that
P says THE TRUTH and Q LIES!

Another Puzzle by Boolean Algebras

P says that "*either P or Q tells the truth*", and
Q says that "*P tells the truth if and only if Q does so*".
Who is lying and who tells the truth?

- P says $P \vee Q$ — Q says $P \leftrightarrow Q$

$P \equiv P \vee Q$ — $Q \equiv P \leftrightarrow Q$

$$\begin{cases} P \equiv P \vee Q \equiv P \vee (P \leftrightarrow Q) \equiv P \vee [(\neg P \vee Q) \wedge (P \vee \neg Q)] \equiv \\ P \vee \neg Q \equiv (P \vee Q) \vee \neg Q \equiv \top. \\ Q \equiv P \leftrightarrow Q \equiv (P \vee Q) \leftrightarrow Q \equiv (P \vee Q) \to Q \equiv P \to Q \ldots \end{cases}$$

(P says $P \vee Q$) and (Q says $P \leftrightarrow Q$) imply that

P says THE TRUTH and Q ??!

Logic Again

- If $\mathscr{A} \to \mathscr{B}$, and if $\mathscr{B}$, then can we infer that $\mathscr{A}$?

  $(\neg a \vee b) \wedge b \equiv b \not\to a$ x
- If $\mathscr{A} \to \mathscr{B}$, and if $\mathscr{A}$, then can we infer that $\mathscr{B}$?

  $(\neg a \vee b) \wedge a \equiv b \wedge a \to b$ ✓
- If $\mathscr{A} \to \mathscr{B}$, and if $\neg\mathscr{B}$, then can we infer that $\neg\mathscr{A}$?

  $(\neg a \vee b) \wedge \neg b \equiv \neg a \wedge \neg b \to \neg a$ ✓
- If $\mathscr{A} \to \mathscr{B}$, and if $\neg\mathscr{A}$, then can we infer that $\neg\mathscr{B}$?

  $(\neg a \vee b) \wedge \neg a \equiv \neg a \not\to \neg b$ x

https://www.wolframalpha.com/

Completeness of Boolean Algebras

Theorem (Completeness)

If $a \equiv b$ is valid according to the truth-table semantics, then it is provable from the axioms. ■

The completeness of Propositional Logic with respect to truth-table semantics follows from Completeness Theorem.

For example, the validity of the formula $[(p \rightarrow q) \rightarrow p] \rightarrow p$, Peirce's Law, can be proved by first translating $a \rightarrow b$ to $\neg a \vee b$, and then showing the equivalence $(\neg[\neg(\neg p \vee q) \vee p] \vee p) \equiv \top$ by the above axioms.

Some Exercises (1)

Three boxes are presented to you.
One contains gold, the other two are empty.
Each box has a message on its door:

Box 1 The gold is not here.

Box 2 The gold is not here.

Box 3 The gold is in Box 2.

Only one message is true; the other two are false.
Which box has the gold?

Some Exercises (2)

In Boolean Algebras define the connective

$$p \rightarrow q \equiv \neg p \vee q$$

Prove that:

- $[a \rightarrow (b \rightarrow a)] \equiv \top$
- $[a \rightarrow (b \rightarrow c)] \equiv [(a \rightarrow b) \rightarrow (a \rightarrow c)]$
- $(\neg b \rightarrow \neg a) \equiv (a \rightarrow b)$

Prove or Disprove:

- $(a \rightarrow b \wedge c) \equiv (a \rightarrow b) \wedge (a \rightarrow c)$
- $(b \vee c \rightarrow a) \equiv (b \rightarrow a) \wedge (c \rightarrow a)$
- $(a \rightarrow b \vee c) \equiv (a \rightarrow b) \vee (a \rightarrow c)$
- $(b \wedge c \rightarrow a) \equiv (b \rightarrow a) \wedge (c \rightarrow a)$

Some Exercises (3)

In Boolean Algebras define the connective

$$p\triangle q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$$

Prove that:

- $(a\triangle a) \equiv \bot$
- $\neg(a\triangle b) \equiv (a\triangle \neg b) \equiv (\neg a\triangle b)$
- $a\triangle(b\triangle c) \equiv (a\triangle b)\triangle c$

Prove or Disprove:

- $a \wedge (b\triangle c) \equiv (a \wedge b)\triangle(a \wedge c)$
- $a \vee (b\triangle c) \equiv (a \vee b)\triangle(a \vee c)$
- $a \rightarrow (b\triangle c) \equiv (a \rightarrow b)\triangle(a \rightarrow c)$
- $a \rightarrow (b\triangle c) \equiv (a \rightarrow b \vee c) \wedge (a \rightarrow \neg b \vee \neg c)$