A Quick Introduction to Mathematical Logic

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Aristotle's Syllogism, 25 August 2021

Shortage of Propositional Logic

This deduction cannot be formalized in Propositional Logic:

$$\frac{\text{All humans are mortal.} \qquad \text{Socrates is a human.}}{\therefore \qquad \text{Socrates is mortal.}}$$

We need some

- predicates for expressing **properties** of subjects,

and some

- quantifiers to **quantify** (the number of) **the subjects** satisfying some properties (in terms of ALL or NONE).

Syllogism

$\mathcal{S}$=subject
$\mathcal{P}$=property

Aristotle's Syllogistic Connectives: $\mathfrak{a}$ $\mathfrak{e}$ $\mathfrak{i}$

- $\mathcal{S}\mathfrak{a}\mathcal{P}$: Every $\mathcal{S}$ is $\mathcal{P}$.
- $\mathcal{S}\mathfrak{i}\mathcal{P}$: Some $\mathcal{S}$ is $\mathcal{P}$.
- $\mathcal{S}\mathfrak{e}\mathcal{P}$: No $\mathcal{S}$ is $\mathcal{P}$. Every $\mathcal{S}$ is *not* $\mathcal{P}$

Later was added: $\mathfrak{o}$

- $\mathcal{S}\mathfrak{o}\mathcal{P}$: Some $\mathcal{S}$ is *not* $\mathcal{P}$.

Syllogistic Rules

$\mathcal{M}$=middle property

First Figure $\dfrac{\mathcal{M}\square\mathcal{P}, \quad \mathcal{S}\bigcirc\mathcal{M}}{\mathcal{S}\triangle\mathcal{P}}$ example : $\dfrac{\mathcal{M}\mathfrak{a}\mathcal{P}, \ \mathcal{S}\mathfrak{a}\mathcal{M}}{\mathcal{S}\mathfrak{a}\mathcal{P}}$

Second Figure $\dfrac{\mathcal{P}\square\mathcal{M}, \quad \mathcal{S}\bigcirc\mathcal{M}}{\mathcal{S}\triangle\mathcal{P}}$ example : $\dfrac{\mathcal{P}\mathfrak{e}\mathcal{M}, \ \mathcal{S}\mathfrak{i}\mathcal{M}}{\mathcal{S}\mathfrak{i}\mathcal{P}}$

Third Figure $\dfrac{\mathcal{M}\square\mathcal{P}, \quad \mathcal{M}\bigcirc\mathcal{S}}{\mathcal{S}\triangle\mathcal{P}}$ example : $\dfrac{\mathcal{M}\mathfrak{e}\mathcal{P}, \ \mathcal{M}\mathfrak{a}\mathcal{S}}{\mathcal{S}\mathfrak{o}\mathcal{P}}$

Fourth Figure $\dfrac{\mathcal{P}\square\mathcal{M}, \quad \mathcal{M}\bigcirc\mathcal{S}}{\mathcal{S}\triangle\mathcal{P}}$ example : $\dfrac{\mathcal{P}\mathfrak{a}\mathcal{M}, \ \mathcal{M}\mathfrak{i}\mathcal{S}}{\mathcal{S}\mathfrak{o}\mathcal{P}}$

Aristotle's Syllogistic *Valid* Rules – First Figure

$$\frac{\mathcal{M}a\mathcal{P}, \quad \mathcal{S}a\mathcal{M}}{\mathcal{S}a\mathcal{P}}$$

$$\frac{\text{all M is P}, \quad \text{all S is M}}{\text{all S is P}}$$

$$\frac{\mathcal{M}a\mathcal{P}, \quad \mathcal{S}i\mathcal{M}}{\mathcal{S}i\mathcal{P}} *$$

$$\frac{\text{all M is P}, \quad \text{some S is M}}{\text{some S is P}}$$

$$\frac{\mathcal{M}e\mathcal{P}, \quad \mathcal{S}a\mathcal{M}}{\mathcal{S}e\mathcal{P}}$$

$$\frac{\text{no M is P}, \quad \text{all S is M}}{\text{no S is P}}$$

$$\frac{\mathcal{M}e\mathcal{P}, \quad \mathcal{S}i\mathcal{M}}{\mathcal{S}o\mathcal{P}}$$

$$\frac{\text{no M is P}, \quad \text{some S is M}}{\text{some S is not P}}$$

Aristotle's Syllogistic *Valid* Rules – Second Figure

$$\frac{\mathcal{P}e\mathcal{M}, \quad \mathcal{S}a\mathcal{M}}{\mathcal{S}e\mathcal{P}} \qquad \frac{\text{no P is M,} \quad \text{all S is M}}{\text{no S is P}}$$

$$\frac{\mathcal{P}a\mathcal{M}, \quad \mathcal{S}e\mathcal{M}}{\mathcal{S}e\mathcal{P}} \qquad \frac{\text{all P is M,} \quad \text{no S is M}}{\text{no S is P}}$$

$$\frac{\mathcal{P}e\mathcal{M}, \quad \mathcal{S}i\mathcal{M}}{\mathcal{S}o\mathcal{P}} \qquad \frac{\text{no P is M,} \quad \text{some S is M}}{\text{some S is not P}}$$

$$\frac{\mathcal{P}a\mathcal{M}, \quad \mathcal{S}o\mathcal{M}}{\mathcal{S}o\mathcal{P}} \qquad \frac{\text{all P is M,} \quad \text{some S is not M}}{\text{some S is not P}}$$

Aristotle's Syllogistic *Valid* Rules – Third Figure

$$\frac{\mathcal{M}a\mathcal{P}, \quad \mathcal{M}a\mathcal{S}}{\mathcal{S}i\mathcal{P}}?$$ $$\frac{\text{all M is P}, \quad \text{all M is S}}{\text{some S is P}}$$

$$\frac{\mathcal{M}i\mathcal{P}, \quad \mathcal{M}a\mathcal{S}}{\mathcal{S}i\mathcal{P}}$$ $$\frac{\text{some M is P}, \quad \text{all M is S}}{\text{some S is P}}$$

$$\frac{\mathcal{M}a\mathcal{P}, \quad \mathcal{M}i\mathcal{S}}{\mathcal{S}i\mathcal{P}}$$ $$\frac{\text{all M is P}, \quad \text{some M is S}}{\text{some S is P}}$$

$$\frac{\mathcal{M}e\mathcal{P}, \quad \mathcal{M}a\mathcal{S}}{\mathcal{S}o\mathcal{P}}?$$ $$\frac{\text{no M is P}, \quad \text{all M is S}}{\text{some S is not P}}$$

$$\frac{\mathcal{M}o\mathcal{P}, \quad \mathcal{M}a\mathcal{S}}{\mathcal{S}o\mathcal{P}}$$ $$\frac{\text{some M is not P}, \quad \text{all M is S}}{\text{some S is not P}}$$

$$\frac{\mathcal{M}e\mathcal{P}, \quad \mathcal{M}i\mathcal{S}}{\mathcal{S}o\mathcal{P}}$$ $$\frac{\text{no M is P}, \quad \text{some M is S}}{\text{some S is not P}}$$

Aristotle's Syllogistic *Valid* Rules – Fourth Figure

$$\frac{\mathcal{P}\mathtt{a}\mathcal{M}, \quad \mathcal{M}\mathtt{a}\mathcal{S}}{\mathcal{S}\mathtt{i}\mathcal{P}}?$$

$$\frac{\text{all P is M,} \quad \text{all M is S}}{\text{some S is P}}$$

$$\frac{\mathcal{P}\mathtt{i}\mathcal{M}, \quad \mathcal{M}\mathtt{a}\mathcal{S}}{\mathcal{S}\mathtt{i}\mathcal{P}}$$

$$\frac{\text{some P is M,} \quad \text{all M is S}}{\text{some S is P}}$$

$$\frac{\mathcal{P}\mathtt{a}\mathcal{M}, \quad \mathcal{M}\mathtt{e}\mathcal{S}}{\mathcal{S}\mathtt{e}\mathcal{P}}$$

$$\frac{\text{all P is M,} \quad \text{no M is S}}{\text{no S is P}}$$

$$\frac{\mathcal{P}\mathtt{e}\mathcal{M}, \quad \mathcal{M}\mathtt{a}\mathcal{S}}{\mathcal{S}\mathtt{o}\mathcal{P}}?$$

$$\frac{\text{no P is M,} \quad \text{all M is S}}{\text{some S is not P}}$$

$$\frac{\mathcal{P}\mathtt{e}\mathcal{M}, \quad \mathcal{M}\mathtt{i}\mathcal{S}}{\mathcal{S}\mathtt{o}\mathcal{P}}$$

$$\frac{\text{no P is M,} \quad \text{some M is S}}{\text{some S is not P}}$$

Syllogism, Set Theoretically

http://www.thefirstscience.org/syllogistic/

Aristotle's Syllogistic Connectives: $\mathfrak{a}$ $\mathfrak{e}$ $\mathfrak{i}$

- $\mathcal{S}\mathfrak{a}\mathcal{P}$: Every $\mathcal{S}$ is $\mathcal{P}$. $\mathcal{S}\subseteq\mathcal{P}$
- $\mathcal{S}\mathfrak{i}\mathcal{P}$: Some $\mathcal{S}$ is $\mathcal{P}$. $\mathcal{S}\cap\mathcal{P}\neq\emptyset$
- $\mathcal{S}\mathfrak{e}\mathcal{P}$: No $\mathcal{S}$ is $\mathcal{P}$. $\mathcal{S}\cap\mathcal{P}=\emptyset$

Later was added: $\mathfrak{o}$

- $\mathcal{S}\mathfrak{o}\mathcal{P}$: Some $\mathcal{S}$ is not $\mathcal{P}$. $\mathcal{S}\nsubseteq\mathcal{P}$

http://www.butte.edu/resources/interim/wmwu//iLogic/2.5/iLogic_2_5.html

Aristotle's Syllogistic *(In)valid* Rules

Some rules are not valid – require non-emptiness conditions.

$$\frac{\mathcal{P}\mathfrak{a}\mathcal{M}, \quad \mathcal{M}\mathfrak{a}\mathcal{S}}{\mathcal{S}\mathfrak{i}\mathcal{P}}\mathcal{P}=\emptyset \qquad \frac{\mathcal{P}\subseteq\mathcal{M}, \quad \mathcal{M}\subseteq\mathcal{S}}{\mathcal{S}\cap\mathcal{P}\neq\emptyset}\text{X}$$

$$\frac{\mathcal{M}\mathfrak{a}\mathcal{P}, \quad \mathcal{M}\mathfrak{a}\mathcal{S}}{\mathcal{S}\mathfrak{i}\mathcal{P}}\mathcal{M}=\emptyset \qquad \frac{\mathcal{M}\subseteq\mathcal{P}, \quad \mathcal{M}\subseteq\mathcal{S}}{\mathcal{S}\cap\mathcal{P}\neq\emptyset}\text{X}$$

$$\frac{\mathcal{P}\mathfrak{e}\mathcal{M}, \quad \mathcal{M}\mathfrak{a}\mathcal{S}}{\mathcal{S}\mathfrak{o}\mathcal{P}}\mathcal{M}=\emptyset \qquad \frac{\mathcal{P}\cap\mathcal{M}=\emptyset, \quad \mathcal{M}\subseteq\mathcal{S}}{\mathcal{S}\nsubseteq\mathcal{P}}\text{X}$$

$$\frac{\mathcal{M}\mathfrak{e}\mathcal{P}, \quad \mathcal{M}\mathfrak{a}\mathcal{S}}{\mathcal{S}\mathfrak{o}\mathcal{P}}\mathcal{M}=\emptyset \qquad \frac{\mathcal{M}\cap\mathcal{P}=\emptyset, \quad \mathcal{M}\subseteq\mathcal{S}}{\mathcal{S}\nsubseteq\mathcal{P}}\text{X}$$