

# Theorems of Tarski and Gödel's Second Incompleteness—Computationally

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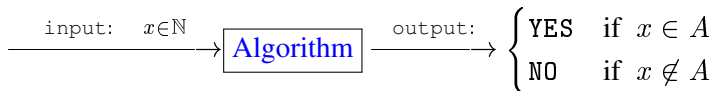
## A Finitely Given Infinite Set

$$\{0, 3, 6, 9, \dots, 3k, \dots\} \subseteq \mathbb{N}$$

$$\{0, 1, 4, 9, \dots, k^2, \dots\} \subseteq \mathbb{N}$$

$$\vdots$$

Computably Decidable set  $A$ : an algorithm  $\mathcal{P}$  decides on any input  $x$  whether  $x \in A$  (outputs YES) or  $x \notin A$  (outputs NO).



**Algorithm:** single-input (natural number), Boolean-output (1, 0)

## A Finitely Given Infinite Set

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$$\vdots$$

Computationally Enumerable set  $A$ : an (input-free) algorithm  $\mathcal{P}$  lists all members of  $A$ ; i.e.,  $A = \text{output}(\mathcal{P})$ .

$$\boxed{\text{Algorithm}} \xrightarrow{\text{output:}} \{a_0, a_1, a_2, \dots\} = A$$

**Algorithm:** input-free, output (a set of natural numbers)

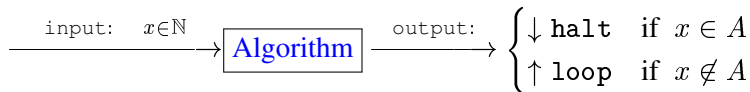
## A Finitely Given Infinite Set

$$\{0, 3, 6, 9, \dots, 3k, \dots\} \subseteq \mathbb{N}$$

$$\{0, 1, 4, 9, \dots, k^2, \dots\} \subseteq \mathbb{N}$$

$$\vdots$$

Semi-Decidable set  $A$ : an algorithm  $\mathcal{P}$  halts on any input  $x$  if and only if  $x \in A$  ( and does not halt if and only if  $x \notin A$  ).



**Algorithm:** single-input (natural number), output-free

## Two Deep Facts from Computability Theory

Semi-Decidable  $\equiv$  Computationally Enumerable (CE)

Decidable  $\equiv$  CE & co-CE

Theorem of Post-Kleene

## A Finitely Given Infinite Set

$$\{0, 3, 6, 9, \dots, 3k, \dots\} \subseteq \mathbb{N}$$

$$\{0, 1, 4, 9, \dots, k^2, \dots\} \subseteq \mathbb{N}$$

$$\vdots$$

**Definable set  $A$ :** a formula  $\varphi(x)$  which holds of  $x$  if and only if  $x \in A$  (and is not true of  $x$  if and only if  $x \notin A$ ).

$$A = \{n \in \mathbb{N} \mid \langle \mathbb{N}; +, \times \rangle \models \varphi(n)\}$$

**Formula:** of the language of arithmetic  $\{+, \times\}$

$$\langle 0, 1, \mathbf{s}, +, \times, \leq, \dots \rangle$$

## Arithmetical Hierarchy of Formulas

 $\neg, \wedge, \vee, \rightarrow$ 

Decidable

 $-\overset{c}{-}, \quad - \cap -, \quad - \cup -, \quad -\overset{c}{\cup} -$ 
 $\exists$     infinite search

 $\forall$     infinite verify

## A Clever Idea

 $\exists x \leq t$     finite search ( $\bigvee_{x \leq t}$ )       $\forall x \leq t$     finite verify ( $\bigwedge_{x \leq t}$ )

## Arithmetical Hierarchy of Formulas

$\Delta_0$  = the class of formulas all whose quantifiers are bounded  
 ( e.g.  $x \in \{0, 1, 4, 9, \dots, k^2, \dots\} \iff \exists y \leq x [x = y^2]$  )

$$\Sigma_1 = \exists v_1 \cdots \exists v_m \Delta_0(v_1, \dots, v_m)$$

$$\Pi_1 = \forall v_1 \cdots \forall v_m \Delta_0(v_1, \dots, v_m)$$

$$\Delta_1 = \Sigma_1 \cap \Pi_1$$

$$\vdots$$

$$\Sigma_{n+1} = \exists v_1 \cdots \exists v_m \Pi_n(v_1, \dots, v_m)$$

$$\Pi_{n+1} = \forall v_1 \cdots \forall v_m \Sigma_n(v_1, \dots, v_m)$$

$$\Delta_n = \Sigma_n \cap \Pi_n$$

$$\vdots$$



## Two Deep Facts from Mathematical Logic

$\Sigma_n$  = closed under  $\wedge, \vee, \forall x \leq t, \exists$

NOT  $\neg, \forall$

$\Pi_n$  = closed under  $\wedge, \vee, \exists x \leq t, \forall$

NOT  $\neg, \exists$

$\Delta_n$  = closed under  $\wedge, \vee, \exists x \leq t, \forall x \leq t, \neg$

NOT  $\forall, \exists$

$\Sigma_1$ -definable (subsets of  $\mathbb{N}$ )  $\equiv$  **CE** (Computably Enumerable)

$\Delta_1$ -definable (subsets of  $\mathbb{N}$ )  $\equiv$  **Computably Decidable**

$\Pi_1$ -definable (subsets of  $\mathbb{N}$ )  $\equiv$  **co-CE**

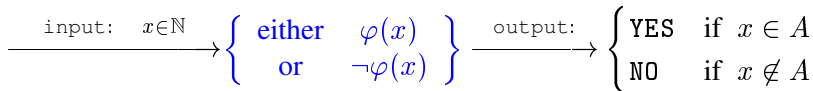
A Motto of Computability Theory (and Mathematical Logic)

**Computability** is **Definability**

A Motto of Mathematical Logic (and Computability Theory)

**Definability** is (Relativized) **Computability** (by Oracles)

$$A = \{u \in \mathbb{N} \mid \langle \mathbb{N}; +, \times \rangle \models \varphi(u/x)\}$$



## Some (Advanced) Higher Recursion Theory

For  $A = \{u \in \mathbb{N} \mid \langle \mathbb{N}; +, \times \rangle \models \varphi(u/x)\}$  if  $\varphi \in \Sigma_n$  then for the Oracle  $\emptyset^{(n)} = \{u \in \mathbb{N} \mid \mathbb{N} \models \Sigma_n\text{-True}(u)\}$  we have

$A \leq_1 \emptyset^{(n)}$  by (the injection)  $f : \mathbb{N} \rightarrow \mathbb{N}$ ,  $f(u) = \ulcorner \varphi(u/x) \urcorner$ :

$$u \in A \iff \mathbb{N} \models \varphi(u/x) \iff \Sigma_n\text{-True}(\ulcorner \varphi(u/x) \urcorner) \iff f(u) \in \emptyset^{(n)}$$

and so  $A \leq_m \emptyset^{(n)}$  and  $A \leq_T \emptyset^{(n)}$  ... etc.

## A Finitely Given (Infinite) Set

Is A Definable Set.

The Complexity of its Definition describes  
the Complexity of its Computation  
(taking an element and determining if it belongs to this set)

## Gödel's First Incompleteness Theorem

in semantic form:

$\text{Th}(\mathbb{N}) = \{\theta \in \text{Sent} \mid \mathbb{N} \models \theta\}$  is Not Decidable.

It is neither CE nor co-CE.

**Proof.**

If  $\text{Th}(\mathbb{N})$  were CE then so would be  $\{\neg\theta \mid \theta \in \text{Th}(\mathbb{N})\} = \text{Th}(\mathbb{N})^c$ ;  
and so  $\text{Th}(\mathbb{N})$  would be decidable!

For the same reason  $\text{Th}(\mathbb{N})$  cannot be co-CE. □

Recall that  $\text{Th}(\mathbb{N})$  is a complete theory!

## Part I: Tarski's Undefinability Theorem

A Reading of the Incompleteness Theorem:

Any CE and sound theory is incomplete

$$T \in \Sigma_1, T \subseteq \text{Th}(\mathbb{N}) \implies T \neq \text{Th}(\mathbb{N})$$

a consequence of  $\text{Th}(\mathbb{N}) \notin \Sigma_1\text{-Definable}$

Tarski's Undefinability Theorem:  $\text{Th}(\mathbb{N}) \notin \text{Definable}$

Corollary of Tarski:  $T \in \Sigma_n, T \subseteq \text{Th}(\mathbb{N}) \implies \text{Th}(\mathbb{N}) \not\subseteq T$

Precise Gödel's 1st:  $T \in \Sigma_1, T \subseteq \text{Th}(\mathbb{N}) \implies \Pi_1\text{-Th}(\mathbb{N}) \not\subseteq T$

Salehi&Seraji (2015):  $T \in \Sigma_n, T \subseteq \text{Th}(\mathbb{N}) \implies \Pi_n\text{-Th}(\mathbb{N}) \not\subseteq T$

$[n = 1]$  ↙ ↘  $[\Pi_n\text{-Th}(\mathbb{N}) \subseteq \text{Th}(\mathbb{N})]$   
 Gödel's 1<sup>st</sup>      Tarski

## Part I: Tarski's Undefinability Theorem

A Unification (and A Generalization for both) of the Theorems of  
**Gödel's 1st Incompleteness** and **Tarski's Undefinability**:

Theorem (**Salehi&Seraji (2015)**)

$T \in \Sigma_n, T \subseteq \text{Th}(\mathbb{N}) \implies \Pi_n\text{-Th}(\mathbb{N}) \not\subseteq T$  (for every  $n > 0$ ).

**Proof.**

If  $T \in \Sigma_n$  then  $\text{Prov}_T \in \Sigma_n$ , and so for the Gödel Sentence  $\gamma$  with  
 $Q \vdash \gamma \iff \neg \text{Pr}_T(\ulcorner \gamma \urcorner)$  we have  $\gamma \in \Pi_n\text{-Th}(\mathbb{N})$  and  $T \not\vdash \gamma$ . □

**So,  $\text{Th}(\mathbb{N})$  Is Not Computable By Any Definable Oracle!**

## Part 2: Gödel's Second Incompleteness Theorem

Some More Technicalities of Gödel's 1st:

- It Is Usually Proved For Peano's Arithmetic PA.

PA is (proved to be [after Gödel]) not finitely axiomatizable.

### A Clever Idea

A Finitely Axiomatizable Arithmetical Theory, called Robinson's Arithmetic  $Q$  Suffices for the Gödel's Arguments to go through ...

### Question

What does  $Q$  in  $Q$  stand for? And what is the theory  $R$ ? Or, possibly  $S$ ?  
Doesn't Robinson Start with  $R$ ? Isn't  $RA = \text{Robinson's Arithmetic}$ ?



## More On Robinson's Arithmetic $Q$

- $Q$  is **finite**:  
 $Q = PA - \{\text{all induction axioms}\} + \forall x \exists y [x = 0 \vee x = S(y)]$
- $Q$  is  $\Sigma_1$ -complete:  $\Sigma_1\text{-Th}(\mathbb{N}) \subseteq Q$ .
- $Q$  is *essentially undecidable*; i.e., **CE incomplete**:  
 every CE and consistent extension of it is incomplete.

So,  $Q$  is **undecidable** (otherwise it could be extended to a consistent, complete and decidable [so CE] theory.)

**Application**: Church's Theorem on the Undecidability of First Order Logic follows from Gödel's 1st Incompleteness Theorem for  $Q$ .

## 2<sup>nd</sup> Application: Gödel's 2<sup>nd</sup> Incompleteness Theorem

Standard (Classic, Usual) Proofs of G2:

Derivability Conditions:

- (i) if  $T \vdash \varphi$  then  $T \vdash \text{Pr}_T(\ulcorner \varphi \urcorner)$
- (ii)  $T \vdash \text{Pr}_T(\ulcorner \varphi \rightarrow \psi \urcorner) \rightarrow [\text{Pr}_T(\ulcorner \varphi \urcorner) \rightarrow \text{Pr}_T(\ulcorner \psi \urcorner)]$
- (iii)  $T \vdash \text{Pr}_T(\ulcorner \varphi \urcorner) \rightarrow \text{Pr}_T(\ulcorner \text{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$

Classically, (iii) is proved by showing:

- (iv)  $T \vdash \sigma \rightarrow \text{Pr}_T(\ulcorner \sigma \urcorner)$  for any  $\sigma \in \Sigma_1$

Usually the following instance of Diagonal Lemma is used:

- (v)  $T \vdash \gamma \leftrightarrow \neg \text{Pr}_T(\ulcorner \gamma \urcorner)$  for some  $\gamma \in \Pi_1$

## Theorem (Gödel's 2nd)

*For any consistent  $T$  satisfying (i,ii,iv,v),  $T \not\vdash \neg\text{Pr}_T(\ulcorner \perp \urcorner)$ .*

### Proof.

By (i) and (v) we have  $T \not\vdash \gamma$ . By (iv),  $T \vdash \neg\gamma \rightarrow \text{Pr}_T(\ulcorner \neg\gamma \urcorner)$ , and so  $(\star) T \vdash \neg\text{Pr}_T(\ulcorner \neg\gamma \urcorner) \rightarrow \gamma$ . By (i), (ii) and classical logic  $T \vdash \text{Pr}_T(\ulcorner \neg\gamma \urcorner) \rightarrow [\text{Pr}_T(\ulcorner \gamma \urcorner) \rightarrow \text{Pr}_T(\ulcorner \perp \urcorner)]$ . Whence,  
 $T \vdash \neg\text{Pr}_T(\ulcorner \perp \urcorner) \rightarrow \neg\text{Pr}_T(\ulcorner \gamma \urcorner) \vee \neg\text{Pr}_T(\ulcorner \neg\gamma \urcorner)$   
by (v)  $\searrow$   $\gamma$   $\swarrow$  by  $(\star)$

And so  $T \vdash \neg\text{Pr}_T(\ulcorner \perp \urcorner) \rightarrow \gamma$ , thus  $T \not\vdash \neg\text{Pr}_T(\ulcorner \perp \urcorner)$ . □

## Gödel's 2<sup>nd</sup> Incompleteness Theorem

It Suffices to Note that:

- (i') if  $U \vdash \varphi$  then  $\mathbb{Q} \vdash \text{Pr}_U(\ulcorner \varphi \urcorner)$  for every  $U \in \Sigma_1$
- (ii')  $\mathbb{N} \models \text{Pr}_U(\ulcorner \varphi \rightarrow \psi \urcorner) \rightarrow [\text{Pr}_U(\ulcorner \varphi \urcorner) \rightarrow \text{Pr}_U(\ulcorner \psi \urcorner)]$  for every  $U$
- (iv')  $\mathbb{N} \models \sigma \rightarrow \text{Pr}_U(\ulcorner \sigma \urcorner)$  for any  $\sigma \in \Sigma_1$  and  $U \supseteq \mathbb{Q}$
- (v)  $\mathbb{Q} \vdash \gamma \iff \neg \text{Pr}_T(\ulcorner \gamma \urcorner)$  for some  $\gamma \in \Pi_1$

$U =$  (Any) Ideal Mathematical Theory

$\mathbb{Q} =$  A Real Mathematical Theory

$\mathbb{Q} \vdash (i'), (v) \quad \text{Real Math. Th.} \vdash (ii'), (iv') \implies$  Failure of Hilbert's Programme

## Part 2: Gödel's Second Incompleteness Theorem

Let

$$\mathbb{Q}' = \mathbb{Q} \cup \{ \text{Pr}_U(\ulcorner \varphi \rightarrow \psi \urcorner) \rightarrow [ \text{Pr}_U(\ulcorner \varphi \urcorner) \rightarrow \text{Pr}_U(\ulcorner \psi \urcorner) ] \mid U \in \Sigma_1 \} \\ \cup \{ \sigma \rightarrow \text{Pr}_U(\ulcorner \sigma \urcorner) \mid \sigma \in \Sigma_1, \mathbb{Q} \subseteq U \in \Sigma_1 \}.$$

Theorem (Salehi — Unpublished)

$$\mathbb{Q}' \in \Sigma_1 \text{ and for any consistent } T, \mathbb{Q}' \subseteq T \in \Sigma_1 \implies T \not\vdash \neg \text{Pr}_T(\ulcorner \perp \urcorner).$$

Gödel's (and Rosser's) 1st Incompleteness Theorem

$$\mathbb{Q} \in \text{Finite and for any consistent } T, \mathbb{Q} \subseteq T \in \Sigma_1 \implies T \notin \Pi_1\text{-Deciding.}$$

A Real Mathematical Theory  $\mathbb{Q}' \vdash (i'), (ii'), (iv'), (v)$

$T \not\vdash \text{Consistency}(U)$  for any real CE  $T \supseteq \mathbb{Q}'$  and ideal CE  $U \supseteq T$

Thank You!

The Participants .....For Listening...

and

The Organizers ....For Taking Care of Everything...

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