Theorems of Tarski and Gödel's Second Incompleteness—Computationally

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A Finitely Given Infinite Set

$$\{0, 3, 6, 9, \cdots, 3k, \cdots\} \subseteq \mathbb{N}$$
$$\{0, 1, 4, 9, \cdots, k^2, \cdots\} \subseteq \mathbb{N}$$
$$\vdots$$

Computably Decidable set A: an algorithm $\mathcal{P}$ decides on any input x whether $x \in A$ (outputs YES) or $x \notin A$ (outputs NO).

input: $x \in \mathbb{N}$ → **Algorithm** → output: $\begin{cases} \texttt{YES} & \text{if } x \in A \\ \texttt{NO} & \text{if } x \notin A \end{cases}$

Algorithm: single–input (natural number), Boolean–output (1, 0)

A Finitely Given Infinite Set

$$\{0, 3, 6, 9, \cdots, 3k, \cdots\} \subseteq \mathbb{N}$$
$$\{0, 1, 4, 9, \cdots, k^2, \cdots\} \subseteq \mathbb{N}$$
$$\vdots$$

Computably Enumerable set A: an (input-free) algorithm $\mathcal{P}$ lists all members of A; i.e., $A = \texttt{output}(\mathcal{P})$.

$$\boxed{\text{Algorithm}} \xrightarrow{\texttt{output:}} \{a_0, a_1, a_2, \cdots\} = A$$

Algorithm: input–free, output (a set of natural numbers)

A Finitely Given Infinite Set

$$\{0, 3, 6, 9, \cdots, 3k, \cdots\} \subseteq \mathbb{N}$$
$$\{0, 1, 4, 9, \cdots, k^2, \cdots\} \subseteq \mathbb{N}$$
$$\vdots$$

Semi–Decidable set A: an algorithm $\mathcal{P}$ halts on any input x if and only if $x \in A$ $\Big($ and does not halt if and only if $x \notin A$ $\Big)$.

$$\xrightarrow{\texttt{input:}\quad x\in\mathbb{N}} \boxed{\text{Algorithm}} \xrightarrow{\texttt{output:}} \begin{cases} \downarrow \texttt{halt} & \text{if } x \in A \\ \uparrow \texttt{loop} & \text{if } x \notin A \end{cases}$$

Algorithm: single–input (natural number), output–free

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Two Deep Facts from Computability Theory

Semi–Decidable ≡ Computably Enumerable (CE)

Decidable ≡ CE & co–CE

Theorem of Post–Kleene

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A Finitely Given Infinite Set

$\{0, 3, 6, 9, \cdots, 3k, \cdots\} \subseteq \mathbb{N}$

$\{0, 1, 4, 9, \cdots, k^2, \cdots\} \subseteq \mathbb{N}$

$\vdots$

Definable set A: a formula $\varphi(x)$ which holds of x if and only if $x \in A$ (and is not true of x if and only if $x \notin A$).

$$A = \{n \in \mathbb{N} \mid \langle \mathbb{N}; +, \times \rangle \models \varphi(n)\}$$

Formula: of the language of arithmetic $\{+, \times\}$

$\langle 0, 1, \mathsf{s}, +, \times, \leqslant, \cdots \rangle$

Arithmetical Hierarchy of Formulas

$\neg, \wedge, \vee, \rightarrow$ Decidable

$$-^{\complement}, \quad -\cap-, \quad -\cup-, \quad -^{\complement}\cup-$$

$\exists$ infinite search $\forall$ infinite verify

A Clever Idea

$\exists\, x \leqslant t$ finite search ($\bigvee_{x\leqslant t}$) $\forall\, x \leqslant t$ finite verify ($\bigwedge_{x\leqslant t}$)

Arithmetical Hierarchy of Formulas

Δ_0= the class of formulas all whose quantifiers are bounded
(e.g. $x \in \{0, 1, 4, 9, \cdots, k^2, \cdots\} \iff \exists y \leqslant x[x = y^2]$)

$\Sigma_1 = \exists v_1 \cdots \exists v_m \; \Delta_0(v_1, \ldots, v_m)$
$\Pi_1 = \forall v_1 \cdots \forall v_m \; \Delta_0(v_1, \ldots, v_m)$

$\Delta_1 = \Sigma_1 \cap \Pi_1$

$\vdots$

$\Sigma_{n+1} = \exists v_1 \cdots \exists v_m \; \Pi_n(v_1, \ldots, v_m)$
$\Sigma_{n+1} = \forall v_1 \cdots \forall v_m \; \Sigma_n(v_1, \ldots, v_m)$

$\Delta_n = \Sigma_n \cap \Pi_n$

$\vdots$

Two Deep Facts from Mathematical Logic

Σ_n = closed under $\wedge, \vee, \forall x \leqslant t, \exists$ NOT $\boxed{\neg}$, $\boxed{\forall}$

Π_n = closed under $\wedge, \vee, \exists x \leqslant t, \forall$ NOT $\boxed{\neg}$, $\boxed{\exists}$

Δ_n = closed under $\wedge, \vee, \exists x \leqslant t, \forall x \leqslant t, \neg$ NOT $\boxed{\forall}$, $\boxed{\exists}$

Σ_1-definable (subsets of $\mathbb{N}$) $\equiv$ CE (Computably Enumerable)
Δ_1-definable (subsets of $\mathbb{N}$) $\equiv$ Computably Decidable
Π_1-definable (subsets of $\mathbb{N}$) $\equiv$ co–CE

A Motto of Computability Theory (and Mathematical Logic)

Computability is Definability

A Motto of Mathematical Logic (and Computability Theory)

Definability is (Relativized) Computability (by Oracles)

$$A = \{u \in \mathbb{N} \mid \langle \mathbb{N}; +, \times \rangle \models \varphi(u/x)\}$$

$$\xrightarrow{\texttt{input:}\quad x\in\mathbb{N}} \left\{ \begin{array}{cc} \text{either} & \varphi(x) \\ \text{or} & \neg\varphi(x) \end{array} \right\} \xrightarrow{\texttt{output:}} \begin{cases} \texttt{YES} & \text{if } x \in A \\ \texttt{NO} & \text{if } x \notin A \end{cases}$$

Some (Advanced) Higher Recursion Theory

For $A=\{u\in\mathbb{N}\mid\langle\mathbb{N};+,\times\rangle\models\varphi(u/x)\}$ if $\varphi\in\Sigma_n$ then for the Oracle $\emptyset^{(n)}=\{u\in\mathbb{N}\mid\mathbb{N}\models\Sigma_n\text{-True}(u)\}$ we have $A\leqslant_1\emptyset^{(n)}$ by (the injection) $f:\mathbb{N}\rightarrow\mathbb{N}$, $f(u)=\ulcorner\varphi(u/x)\urcorner$:

$$u\in A\iff\mathbb{N}\models\varphi(u/x)\iff\Sigma_n\text{-True}(\ulcorner\varphi(u/x)\urcorner)\iff f(u)\in\emptyset^{(n)}$$

and so $A\leqslant_m\emptyset^{(n)}$ and $A\leqslant_{\mathrm{T}}\emptyset^{(n)}$ $\cdots$ etc.

A Finitely Given (Infinite) Set

Is A Definable Set.

The Complexity of its Definition describes
the Complexity of its Computation
(taking an element and determining if it belongs to this set)

Gödel's First Incompleteness Theorem

in semantic form:

$\mathrm{Th}(\mathbb{N}) = \{\theta \in \mathsf{Sent} \mid \mathbb{N} \models \theta\}$ is Not Decidable.

It is neither CE nor co–CE.

Proof.

If $\mathrm{Th}(\mathbb{N})$ were CE then so would be $\{\neg\theta \mid \theta \in \mathrm{Th}(\mathbb{N})\} = \mathrm{Th}(\mathbb{N})^{\complement}$;
and so $\mathrm{Th}(\mathbb{N})$ would be decidable!
For the same reason $\mathrm{Th}(\mathbb{N})$ cannot be co–CE. □

Recall that $\mathrm{Th}(\mathbb{N})$ is a complete theory!

Part I: Tarski's Undefinability Theorem

A Reading of the Incompleteness Theorem:

Any CE and sound theory is incomplete

$T \in \Sigma_1, T \subseteq \mathrm{Th}(\mathbb{N}) \Longrightarrow T \neq \mathrm{Th}(\mathbb{N})$

a consequence of $\mathrm{Th}(\mathbb{N}) \notin \Sigma_1$-Definable

Tarski's Undefinability Theorem: $\mathrm{Th}(\mathbb{N}) \notin$ Definable

Corollary of Tarski: $T \in \Sigma_n, T \subseteq \mathrm{Th}(\mathbb{N}) \Longrightarrow \mathrm{Th}(\mathbb{N}) \not\subseteq T$

Precise Gödel's 1st: $T \in \Sigma_1, T \subseteq \mathrm{Th}(\mathbb{N}) \Longrightarrow \Pi_1\text{-}\mathrm{Th}(\mathbb{N}) \not\subseteq T$

Salehi&Seraji (2015): $T \in \Sigma_n, T \subseteq \mathrm{Th}(\mathbb{N}) \Longrightarrow \Pi_n\text{-}\mathrm{Th}(\mathbb{N}) \not\subseteq T$

$[n=1] \swarrow \quad \searrow [\Pi_n\text{-}\mathrm{Th}(\mathbb{N}) \subseteq \mathrm{Th}(\mathbb{N})]$

Gödel's 1^{st} Tarski

Part I: Tarski's Undefinability Theorem

A Unification (and A Generalization for both) of the Theorems of
Gödel's 1st Incompleteness and Tarski's Undefinability:

Theorem (Salehi&Seraji (2015))

$T \in \Sigma_n, T \subseteq \mathrm{Th}(\mathbb{N}) \Longrightarrow \Pi_n\text{-}\mathrm{Th}(\mathbb{N}) \not\subseteq T$ (for every $n > 0$).

Proof.

If $T \in \Sigma_n$ then $\mathsf{Prov}_T \in \Sigma_n$, and so for the Gödel Sentence γ with $\mathtt{Q} \vdash \gamma \longleftrightarrow \neg\mathsf{Pr}_T(\ulcorner\gamma\urcorner)$ we have $\gamma \in \Pi_n\text{-}\mathrm{Th}(\mathbb{N})$ and $T \nvdash \gamma$. □

So, $\mathrm{Th}(\mathbb{N})$ Is Not Computable By Any Definable Oracle!

Part 2: Gödel's Second Incompleteness Theorem

Some More Technicalities of Gödel's 1st:

- It Is Usually Proved For Peano's Arithmetic $\mathtt{PA}$.

  $\mathtt{PA}$ is (proved to be [after Gödel]) not finitely axiomatizable.

A Clever Idea

A Finitely Axiomatizable Arithmetical Theory, called Robinson's Arithmetic $\mathtt{Q}$ Suffices for the Gödel's Arguments to go through ...

Question

What does Q in $\mathtt{Q}$ stand for? And what is the theory $\mathtt{R}$? Or, possibly $\mathtt{S}$? Doesn't Robinson Start with $\mathtt{R}$? Isn't $\mathtt{RA}$ = Robinson's Arithmetic?

More On Robinson's Arithmetic Q

- Q is finite:
  $\mathtt{Q} = \mathtt{PA} - \{\text{all induction axioms}\} + \forall x \exists y [x = 0 \vee x = S(y)]$
- Q is Σ_1-complete: $\Sigma_1\text{-Th}(\mathbb{N}) \subseteq \mathtt{Q}$.
- Q is *essentially undecidable*; i.e., CE incompletable:
  every CE and consistent extension of it is incomplete.

So, Q is undecidable (otherwise it could be extended to a consistent, complete and decidable [so CE] theory.)

Application: Church's Theorem on the Undecidability of First Order Logic follows from Gödel's 1st Incompleteness Theorem for Q.

2^{nd} Application: Gödel's 2^{nd} Incompleteness Theorem

Standard (Classic, Usual) Proofs of G2:

Derivability Conditions:

(i) if $T \vdash \varphi$ then $T \vdash \mathsf{Pr}_T(\ulcorner \varphi \urcorner)$

(ii) $T \vdash \mathsf{Pr}_T(\ulcorner \varphi \rightarrow \psi \urcorner) \rightarrow \big[\mathsf{Pr}_T(\ulcorner \varphi \urcorner) \rightarrow \mathsf{Pr}_T(\ulcorner \psi \urcorner)\big]$

(iii) $T \vdash \mathsf{Pr}_T(\ulcorner \varphi \urcorner) \rightarrow \mathsf{Pr}_T(\ulcorner \mathsf{Pr}_T(\ulcorner \varphi \urcorner) \urcorner)$

Classically, (iii) is proved by showing:

(iv) $T \vdash \sigma \rightarrow \mathsf{Pr}_T(\ulcorner \sigma \urcorner)$ for any $\sigma \in \Sigma_1$

Usually the following instance of Diagonal Lemma is used:

(v) $T \vdash \gamma \longleftrightarrow \neg \mathsf{Pr}_T(\ulcorner \gamma \urcorner)$ for some $\gamma \in \Pi_1$

Theorem (Gödel's 2nd)

For any consistent T satisfying (i,ii,iv,v), $T \nvdash \neg \mathrm{Pr}_T(\ulcorner \bot \urcorner)$.

Proof.

By (i) and (v) we have $T \nvdash \gamma$. By (iv), $T \vdash \neg\gamma \rightarrow \mathrm{Pr}_T(\ulcorner \neg\gamma \urcorner)$, and so ($\star$) $T \vdash \neg\mathrm{Pr}_T(\ulcorner \neg\gamma \urcorner) \rightarrow \gamma$. By (i), (ii) and classical logic $T \vdash \mathrm{Pr}_T(\ulcorner \neg\gamma \urcorner) \rightarrow \big[\mathrm{Pr}_T(\ulcorner \gamma \urcorner) \rightarrow \mathrm{Pr}_T(\ulcorner \bot \urcorner)\big]$. Whence,

$$T \vdash \neg\mathrm{Pr}_T(\ulcorner \bot \urcorner) \rightarrow \neg\mathrm{Pr}_T(\ulcorner \gamma \urcorner) \vee \neg\mathrm{Pr}_T(\ulcorner \neg\gamma \urcorner)$$

by (v) $\searrow$ γ $\swarrow$ by ($\star$)

And so $T \vdash \neg\mathrm{Pr}_T(\ulcorner \bot \urcorner) \rightarrow \gamma$, thus $T \nvdash \neg\mathrm{Pr}_T(\ulcorner \bot \urcorner)$. □

Gödel's 2^{nd} Incompleteness Theorem

It Suffices to Note that:

(i') if $U \vdash \varphi$ then $\mathtt{Q} \vdash \mathsf{Pr}_U(\ulcorner \varphi \urcorner)$ for every $U \in \Sigma_1$

(ii') $\mathbb{N} \models \mathsf{Pr}_U(\ulcorner \varphi \rightarrow \psi \urcorner) \rightarrow [\mathsf{Pr}_U(\ulcorner \varphi \urcorner) \rightarrow \mathsf{Pr}_U(\ulcorner \psi \urcorner)]$ for every U

(iv') $\mathbb{N} \models \sigma \rightarrow \mathsf{Pr}_U(\ulcorner \sigma \urcorner)$ for any $\sigma \in \Sigma_1$ and $U \supseteq \mathtt{Q}$

(v) $\mathtt{Q} \vdash \gamma \longleftrightarrow \neg\mathsf{Pr}_T(\ulcorner \gamma \urcorner)$ for some $\gamma \in \Pi_1$

U = (Any) Ideal Mathematical Theory

$\mathtt{Q}$ = A Real Mathematical Theory

$\mathtt{Q} \vdash$(i'),(v) Real Math. Th. $\vdash$ (ii'),(iv') $\Longrightarrow$ Failure of Hilbert's Programme

Part 2: Gödel's Second Incompleteness Theorem

Let

$$\mathfrak{Q}' = \mathtt{Q} \cup \{\mathtt{Pr}_U(\ulcorner \varphi \to \psi \urcorner) \to [\mathtt{Pr}_U(\ulcorner \varphi \urcorner) \to \mathtt{Pr}_U(\ulcorner \psi \urcorner)] \mid U \in \Sigma_1\}$$
$$\cup \{\sigma \to \mathtt{Pr}_U(\ulcorner \sigma \urcorner) \mid \sigma \in \Sigma_1, \mathtt{Q} \subseteq U \in \Sigma_1\}.$$

Theorem (Salehi — Unpublished)

$\mathfrak{Q}' \in \Sigma_1$ and for any consistent T, $\mathfrak{Q}' \subseteq T \in \Sigma_1 \implies T \nvdash \neg \mathrm{Pr}_T(\ulcorner \bot \urcorner)$.

Gödel's (and Rosser's) 1st Incompleteness Theorem

$\mathtt{Q} \in$ Finite and for any consistent T, $\mathtt{Q} \subseteq T \in \Sigma_1 \implies T \notin \Pi_1$-Deciding.

A Real Mathematical Theory $\mathfrak{Q}' \vdash$(i'),(ii'),(iv'),(v)

$T \nvdash \mathsf{Consistency}(U)$ for any real CE $T \supseteq \mathfrak{Q}'$ and ideal CE $U \supseteq T$

Thank You!

The Participants .... For Listening...

and

The Organizers .... For Taking Care of Everything...

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