On Heuristic Principles for Arithmetical Theories

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Chaitin's HP

- On Chaitin's Heuristic Principle and Halting Probability.
  arXiv:2310.14807v3 [math.LO].
  https://arxiv.org/abs/2310.14807

1. **H**euristic **P**rinciple
2. **H**alting **P**robability

► **H**euristic **P**rinciple: Greater Complexity Implies Unprovability

If a sentence is more complex (heavier) than the theory,
then that sentence is *unprovable* from that theory.

Solomonoff-Kolmogorov-Chaitin Complexity

Definition (Program Size Complexity)

$\mathcal{C}(x) =$ the length of
the shortest input-free program that outputs only x (and halts).

Example

$(10)^M = 1010 \cdots 10$	$\{10^n\}_{n=1}^{\infty} = 10100100010000 \cdots 10^n 10^{n+1} \cdots$

BEGIN
  input M
  for $i = 1$ to M
    print **1**
    print **0**
END
	BEGIN
	  let $n = 1$
	  while $n > 0$ do
	    begin
	      print **1**
	      for $i = 1$ to n
	        print **0**
	      let $n = n+1$
	    end
	END

Descriptive Complexity & Randomness

- 111111111111111111111111111111111111 ⋯ 1^*
- 100100100100100100100100100100100 ⋯ $(100)^*$
- 0101101110111101111101111110111 ⋯ $\{01^n\}_{n>0}$
- 0101111010111110111111111011 ⋯ $\{01^{(\pi-3)_n}\}_{n=1}^{\infty}$
- 11000110000111111000010010100001101010 ⋯

Definition (Random)

A random number or a string is one whose program-size complexity is almost its length.

Complexity of Sentences and Theories

Arithmetic:

- $\exists x,y,z\,(xyz\neq 0 \ \wedge\ x^2+y^2=z^2)$ $x{=}3, y{=}4, z{=}5$
- $\neg\exists x,y,z\,(xyz\neq 0 \ \wedge\ x^3+y^3=z^3)$
- $\neg\exists x,y,z\,(xyz\neq 0 \ \wedge\ x^4+y^4=z^4)$
- $\forall n>2\neg\exists x,y,z\,(xyz\neq 0 \ \wedge\ x^n+y^n=z^n)$

Geometry:

- $\forall\triangle ABC\,(M_a,M_b,M_c\text{midpoints}\longrightarrow\exists\mathbb{G}[AM_a\cap BM_b\cap CM_c=\{\mathbb{G}\}])$
- $\forall\triangle ABC\,(AA',BB',CC'\text{altitudes}\longrightarrow\exists\mathbb{H}[AA'\cap BB'\cap CC'=\{\mathbb{H}\}])$
- $\forall\triangle ABC\,\exists!\mathbb{O}\,(\overline{\mathbb{O}A}=\overline{\mathbb{O}B}=\overline{\mathbb{O}C})$
- $\forall\triangle ABC\,(\mathbb{G},\mathbb{H},\mathbb{O}\text{ are identical or on a line})$ (never form a triangle)

Heuristic Principle, HP

Definition (HP-satisfying weighing)

A mapping $\mathscr{W}$ from theories and sentences to $\mathbb{R}$ satisfies HP when, for every theory $\mathcal{T}$ and every sentence ψ we have

$$\mathscr{W}(\psi) > \mathscr{W}(\mathcal{T}) \Longrightarrow \mathcal{T} \nvdash \psi.$$

Equivalently, $\mathcal{T} \vdash \psi \Longrightarrow \mathscr{W}(\mathcal{T}) \geqslant \mathscr{W}(\psi)$

- Chaitin's Idea: program-size complexity
- Lots of Criticisms ...
- Some built their own *partial* weighting
- Fans come to rescue ...

A Fanfare

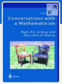
Lecture — Undecidability & Randomness in Pure Mathematics

Gregory J. Chaitin

Chapter

Abstract

I have shown that God not only plays dice in physics, but even in pure mathematics, in elementary number theory, in arithmetic! My work is a fundamental extension of the work of Gödel and Turing on undecidability in pure mathematics. I show that not only does undecidability occur, but in fact sometimes there is complete randomness, and mathematical truth becomes a perfect coin toss.



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Conversations with a Mathematician

Math, Art, Science and the Limits of Reason

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Authors: Gregory J. Chaitin

Written by the author of the best-selling trilogy "The Limits of Mathematics" "The Unknowable" and "Exploring Randomness"

A collection of interviews with Greg Chaitin, the creator of Algorithmic Information Theory

https://doi.org/10.1007/978-1-4471-0185-7_8

Exaggerations and Criticisms

1978: M. Davis: "Chaitin…showed how…to obtain a dramatic extension of Gödel's incompleteness theorem" (*What is a Computation?*, p. 265)

1986: G. Chaitin: "This [the CIT] is a dramatic extension of Gödel's theorem" (*Randomness and Gödel's theorem*, p. 68[Inf.Rand.Inc.$_{1987}$])

1988: I. Stewart: "Chaitin…has proved the ultimate in undecidability theorems…that the logical structures of arithmetic can be random" (*The Ultimate in Undecidability*, **Nature**$_{332}$, p. 115)

1989: G. Chaitin: "I have shown that God…plays dice…in pure math… My work is a fundamental extension of the work of Gödel and Turing on undecid. in pure math" (*Undecidability & Randomness in Pure Math*)

1989: M. van Lambalgen, *Algorithmic Information Theory*, **JSL** 54_4:1389–400.

1996: D. Fallis, *The Source of Chaitin's Incorrectness*, **Phil.Math.III** 4_3:261–96.

1998: P. Raatikainen, *On Interpreting Chaitin's Incom. Thm.*, **JPL** 27_6:569–86.

2000: P. Raatikainen, *Algor. Info. Theory & Undecid.*, **Synthese** 123_2:217–25.

HP, A LOST PARADISE

- CRITICISMS:

  For complex sentences $\mathfrak{S}, \mathfrak{S}'$, or complex numbers $\mathcal{N}, \mathcal{N}'$, the following *complicated* sentences are all provable:

  - $\mathfrak{S} \rightarrow \mathfrak{S}$, $\mathfrak{S} \wedge \mathfrak{S}' \rightarrow \mathfrak{S}' \wedge \mathfrak{S}$, $(\neg\mathfrak{S}' \rightarrow \neg\mathfrak{S}) \Rightarrow (\mathfrak{S} \rightarrow \mathfrak{S}')$.
  - $1+\mathcal{N} = \mathcal{N}+1$, $\mathcal{N} \times \mathcal{N}' = \mathcal{N}' \times \mathcal{N}$, $n(\mathcal{N}+\mathcal{N}') = n\mathcal{N}+n\mathcal{N}'$.

- A SALVAGE?

  △ δ-complexity: $\mathcal{C}(x) - |x|$.
  XXX $\mathcal{T} \vdash \psi \Longrightarrow \delta(\mathcal{T}) \geqslant \delta(\psi)$ XXX

- NO HOPE:

  ▷ $\bot \rightarrow \mathfrak{S}$, $\mathfrak{S} \rightarrow \top$, $p \rightarrow (\mathfrak{S} \rightarrow p)$, $\neg p \rightarrow (p \rightarrow \mathfrak{S})$.
  ▷ $\mathcal{N} > 0$, $\mathcal{N} \times 0 = 0$, $1+\mathcal{N} \neq 1$, $2 \leqslant 2 \times \mathcal{N}$.

HP^{-1}, the converse of HP

$$\mathrm{HP}: \quad \mathcal{T} \vdash \psi \Longrightarrow \mathscr{W}(\mathcal{T}) \geqslant \mathscr{W}(\psi)$$

can be satisfied by any constant weighing.

$$\mathrm{HP}^{-1}: \quad \mathscr{W}(\mathcal{T}) \geqslant \mathscr{W}(\psi) \Longrightarrow \mathcal{T} \vdash \psi$$

cannot hold for real-valued weights since every two real numbers are comparable ($a \geqslant b \vee b \geqslant a$), while some theories and sentences are incomparable, such as ψ and $\neg\psi$ for a non-provable and non-refutable ψ (like any atom in PL or $\forall x \forall y(x = y)$ in FOL).

Both HP and HP^{-1} hold for some non-real-valued weightings.

EP, The Equivalence Principle

$$\text{EP}: \quad \mathscr{W}(\mathcal{T}) = \mathscr{W}(\mathcal{U}) \Longrightarrow \mathcal{T} \equiv \mathcal{U}$$

is a (weak) consequence of HP^{-1}.

This is compatible with HP, even for real-valued weighings.

Theorem (Existence)

There exist some real-valued weightings that satisfy both HP and EP.

Theorem (Un/Computability)

No computable HP+EP-satisfying weighing exists for undecidable logics.
For decidable logics, there are computable HP+EP-satisfying weightings.

The Proof

Definition (Sequence of Sentences)

Let $\psi_1, \psi_2, \psi_3, \cdots$ be an effective list of all the sentences.
For a theory T and $n > 0$, let

$$\chi_n(T) = \begin{cases} 0, & \text{if } T \nvdash \psi_n; \\ 1, & \text{if } T \vdash \psi_n. \end{cases}$$

Finally, let $\mathcal{V}(T) = \sum_{n>0} 2^{-n}\chi_n(T)$.

The Main Observation

For all theories T and U, we have $T \vdash U \iff \forall n > 0\colon \chi_n(T) \geqslant \chi_n(U)$.

$\mathrm{HP} + \mathrm{HP}^{-1}$

So, we have both

$$\mathrm{HP}: T \vdash U \Longrightarrow \mathcal{V}(T) \geqslant \mathcal{V}(U)$$
$$\mathrm{EP}: \mathcal{V}(T) = \mathcal{V}(U) \Longrightarrow T \equiv U$$

Chaitin's Halting Probability

- Halting Probability (of a randomly given input-free program)

$$\Omega = \sum_{p \text{ halts}} 2^{-|p|}.$$

Halting or Looping forever:

A random $\{0,1\}$-string may not be (the ASCII code of) a program.
Even if it is, then it may not be input-free.
If a binary string is (the code of) an input-free program, then
it may halt after running or may loop forever.

$$\Omega = \sum_{p\in\{0,1\}^{+} \text{ halts}}^{p\text{: input-free}} 2^{-|p|}.$$

A Partial Agreement

The probability of getting a fixed binary string of length n by tossing a fair coin (whose one side is '0' and the other '1') is 2^{-n}, and the halting probability of programs with size n is

$$\frac{\text{the number of } \textit{halting programs} \text{ with size } n}{\text{the number of } \textit{all binary strings} \text{ with size } n} = \frac{\#\{p \in \mathbb{P} \colon p\!\downarrow \;\&\; |p| = n\}}{2^n}$$

since there are 2^n binary strings of size n. Thus, the halting probability of programs with size n can be written as $\sum_{p\downarrow}^{|p|=n} 2^{-|p|}$.

Denote this number by Ω_n; so, the number of halting programs with size n is $2^n\Omega_n$.

AND A DISAGREEMENT

According to Chaitin (and almost everybody else), the halting probability of programs with size $\leqslant N$ is $\sum_{n=1}^{N}\Omega_n = \sum_{p\downarrow}^{|p|\leqslant N} 2^{-|p|}$; and so, the halting probability is $\sum_{n=1}^{\infty}\Omega_n = \sum_{p\downarrow} 2^{-|p|} (= \mathbf{\Omega})$!

Let us see why we believe this to be an error. The halting probability of programs with size $\leqslant N$ is in fact

$$\frac{\text{the number of halting programs with size} \leqslant N}{\text{the number of all binary strings with size} \leqslant N} = \frac{\sum_{n=1}^{N} 2^n\Omega_n}{\sum_{n=1}^{N} 2^n}.$$

Now, it is a calculus exercise to notice that, for sufficiently large Ns,

$$\frac{\sum_{n=1}^{N} 2^n\Omega_n}{\sum_{n=1}^{N} 2^n} \neq \sum_{n=1}^{N}\Omega_n, \text{ and } \lim_{N\to\infty}\frac{\sum_{n=1}^{N} 2^n\Omega_n}{\sum_{n=1}^{N} 2^n} \neq \lim_{N\to\infty}\sum_{n=1}^{N}\Omega_n.$$

Ω Is the Probability of Something Else

Conditional Probability

Let $\Omega_S = \sum_{s \in S} 2^{-|s|}$ and $\mho_S = \Omega_S / \Omega_{\mathbb{P}}$ for a set $S \subseteq \mathbb{P}$ of programs.
This is a probability measure: $\mho_\emptyset = 0$, $\mho_{\mathbb{P}} = 1$, and for any family $\{S_i \subseteq \mathbb{P}\}_i$ of pairwise disjoint sets of programs, $\mho_{\bigcup_i S_i} = \sum_i \mho_{S_i}$.
If $\mathcal{H}$ is the set of all the binary codes of the halting programs, then the (conditional) halting probability is $\mho_{\mathcal{H}}$, or $\mathbf{\Omega}/\Omega_{\mathbb{P}}$.
We then have $\mho_{\mathcal{H}} > \mathbf{\Omega}$ since it can be shown that $\Omega_{\mathbb{P}} < 1$.

Theorem (known in the literature)

Chaitin's $\mathbf{\Omega}$ is the probability that the unique infinite binary expansion after $0.$ *of a randomly given real $\alpha \in (0, 1]$ contains a finite binary strings as a prefix that is the binary code of a halting input-free program.*

Halting Probability is NOT Ω !

Asymptotic Probability

Count $\hbar_n$ the number of halting programs (the strings that code some input-free programs that eventually halt after running) that have integer codes‡ less than or equal to n. Then define the halting probability to be $\lim_{n\to\infty} \hbar_n/n$, of course, if it exists. Or take $\lim_{N\to\infty} (\sum_{n=1}^{N} 2^n\Omega_n)/(\sum_{n=1}^{N} 2^n)$ if the limit exists.

Note that this number can be shown to be $\leqslant \dfrac{\mathbf{\Omega}}{2}$.

‡ integer code: $0_1, 1_2, 00_3, 01_4, 10_5, 11_6, 000_7, 001_8, 010_9, \cdots$

Theorem (NEW)

The halting probability of input-free programs is less than $\mathbf{\Omega}$ *under any probability measure on the finite binary strings* (over a fair coin).§

§ $p(0)=p(1),\ p(00)=p(01)=p(10)=p(11),\cdots\ \&\ \sum_{\sigma\in\{0,1\}^+} p(\sigma)=1.$

Thank You!

Thanks to

The Participants .... For Listening ···

and

The Organizer, For Taking Care of Everything ···