

NOTE



## Graph-theoretic proofs of Cantor's power set theorem

Saeed Salehi 

Harish and Bina School of AI and Computer Science, Plaksha University, Mohali, India

### 1. Introduction: diagonal argument

The study of infinite sets begins perhaps with the equinumerosity of some obviously different sets, which can be as astonishing as the existence of a bijective (1–1 and onto) function between the set of natural numbers  $\mathbb{N}$  and the set of all rational numbers (fractions of integers); see e.g. [1, Chapter 19, Theorem 1]. Countable sets, i.e. the sets with a bijective correspondence with  $\mathbb{N}$ , abound in mathematics, from the set of all finite binary strings to the set of all polynomials with rational coefficients. A second shock then shows up: there exists an uncountably infinite set, such as the set of all real numbers; see [1, Chapter 19, Theorem 2]. So, we cannot speak of *the infinite* in mathematics in the same way as in theology or ontology. There are at least two different kinds of infinity: countably infinite sets, like  $\mathbb{N}^*$ —the set of all finite sequences of natural numbers, and uncountably infinite sets, like  $\mathbb{N}^\omega$ —the set of all infinite sequences of natural numbers. The last surprise blow comes from Cantor's power set theorem: infinite sets do not end there, as for any arbitrary set  $S$ , there is another set that  $S$  cannot cover surjectively; such a set is  $\mathcal{P}(S)$ , the power set of  $S$ , the set of all the subsets of  $S$  (see [1, Chapter 19, Appendix]).

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**CONTACT** Saeed Salehi  saesal@gmail.com  Plaksha University, IT City Road, Sector 101A, Mohali, Punjab 140306, India

**Theorem 1 (Cantor’s power set theorem, 1891):** For a given set  $S$ , there is no surjective (onto) function from  $S$  to  $\mathcal{P}(S)$ .

**Proof:** For any function  $\mathcal{F}: S \rightarrow \mathcal{P}(S)$ , the anti-diagonal set  $\mathcal{D} = \{s \in S \mid s \notin \mathcal{F}(s)\}$  is out of  $\mathcal{F}$ ’s range, since  $\mathcal{D} = \mathcal{F}(\sigma)$ , for any  $\sigma \in S$ , would imply  $\sigma \in \mathcal{F}(\sigma) \leftrightarrow \sigma \notin \mathcal{F}(\sigma)$ , a contradiction.

This neat and tidy proof is now called Cantor’s Diagonal Argument.<sup>1</sup> The name comes from the matrix interpretation of the proof. The adjacency matrix of  $\mathcal{F}: S \rightarrow \mathcal{P}(S)$  is an  $S \times S$  matrix, where the columns indicate the members of  $S$  and the rows indicate the values of  $\mathcal{F}$ . Thus, an entry  $(i, j)$  is 1 if  $j \in \mathcal{F}(i)$ , and it is 0 otherwise, when  $j \notin \mathcal{F}(i)$ . Cantor’s proof is called *diagonal* since the complement of the diagonal set is out of  $\mathcal{F}$ ’s range. For example, the diagonal of Table 1 is 10110 . . . , which agrees so far with the row of  $\mathcal{F}(c)$ , and its anti-diagonal is 01001 . . . , which cannot be equal to any row; this indicates  $\{b, e, . . .\}$ , a subset of  $S$  outside  $\mathcal{F}$ ’s range.

**Table 1.** Diagonal = 10110... Anti-Diagonal = 01001... indicating the subset  $\{b, e, . . .\}$ .

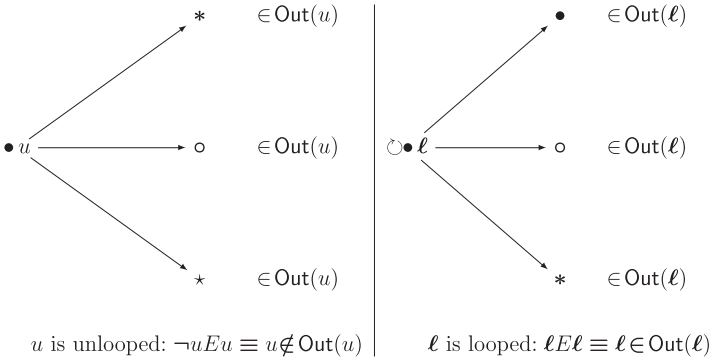
$\mathcal{F} \setminus S$	$a$	$b$	$c$	$d$	$e$	...	
$\mathcal{F}(a)$	<u>1</u>	0	0	0	1	...	$= \{a, b, c, d, e, \dots\}$
$\mathcal{F}(b)$	0	<u>0</u>	1	0	0	...	$= \{a, b, c, d, e, \dots\}$
$\mathcal{F}(c)$	1	0	<u>1</u>	1	0	...	$= \{a, b, c, d, e, \dots\}$
$\mathcal{F}(d)$	1	1	<u>1</u>	<u>1</u>	1	...	$= \{a, b, c, d, e, \dots\}$
$\mathcal{F}(e)$	0	1	0	0	<u>0</u>	...	$= \{a, b, c, d, e, \dots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

This proof, though short and quite elementary, is not the only proof. In this note, we will see a couple of alternative proofs. We will present a graph-theoretic interpretation of Cantor’s diagonal argument, which ‘makes the result more accessible and easy to visualise’, as one referee of this article put it. The graph theory perspective allows us to study a couple of old and new proofs; indeed, ‘the visualisation one gains from considering the argument in terms of graphs and out-sets is very helpful and it might be useful to . . . anybody wanting to teach . . . a similar proof’, as another referee of this article reported. This graph-theoretic visualisation of Cantor’s power set theorem seems new, and the arguments below are the author’s rewording of the old proofs.

## 2. From a graph theory perspective

### 2.1. Definitions

A *directed graph* is a pair  $(V, E)$ , where  $V$  is a nonempty set of vertices and  $E$  is a binary relation of edges between the vertices, so that  $E \subseteq V^2$ . Here,  $vEw$  denotes the fact that an edge in  $E$  connects the vertex  $v \in V$  to the vertex  $w \in V$ , and  $\neg vEw$  indicates that  $v$  is not connected to  $w$  by any edge in  $E$ . When  $vEw$ , we say that  $v$  is *E-connected* to  $w$ . By an *E-path*, we mean a sequence  $w_1, \dots, w_n$  of vertices, possibly with repetition, such that  $w_1E \dots Ew_n$ . The *outgoing set* of a vertex  $v \in V$  is the set of all the vertices to which  $v$  is *E-connected*, i.e.  $\text{Out}(v) = \{w \in V \mid vEw\}$ . A vertex  $v \in V$  is called *looped* when it is connected to itself by an edge in  $E$ , i.e. when  $vEv$ ; otherwise, it is called *unlooped* (when  $\neg vEv$ ).



### 2.2. Proofs

#### 2.2.1. Cantor's original diagonal argument

If a combinatorist, or an AI tool, is asked *why there should exist a set of vertices that is not the outgoing set of any vertex*, the most likely answer will be that *because the subsets of  $V$  outnumber all the outgoing sets*, which are at most as many as the number of all vertices. Indeed, if  $V$  has  $n$  elements, then there are  $2^n$  subsets of  $V$ , and there are at most  $n$  outgoing sets, and since  $2^n > n$ , then, by the pigeonhole principle, there should exist a set of vertices that is not the outgoing set of any vertex. If the graph is infinite, then the argument remains the same; just the 'cardinality' replaces the 'number of

elements'. The ingenious diagonal argument of Cantor can be translated to explicitly constructing a set of vertices that is not equal to any outgoing set. Here is a rephrasing of the diagonal proof.

**Theorem 2 (Unlooped vertices do not make an outgoing set):**

*The set  $\mathcal{D} = \{u \in V \mid \neg uEu\}$  of unlooped vertices is not the outgoing set of any vertex in a directed graph  $(V, E)$ .*

**Proof:** Fix a vertex  $v \in V$ . If  $vEv$ , then  $v \in \text{Out}(v)$  and  $v \notin \mathcal{D}$ , so  $v \in \text{Out}(v) \setminus \mathcal{D}$ . If  $\neg vEv$ , then  $v \notin \text{Out}(v)$  and  $v \in \mathcal{D}$ , so  $v \in \mathcal{D} \setminus \text{Out}(v)$ . Thus, we always have  $v \in [\text{Out}(v) \setminus \mathcal{D}] \cup [\mathcal{D} \setminus \text{Out}(v)]$ . Therefore,  $\mathcal{D} \neq \text{Out}(v)$  for each and every vertex  $v \in V$ .

This goes to say that the outgoing operation,  $\text{Out}: V \rightarrow \mathcal{P}(V)$ , is not surjective. To see that no function  $\mathcal{F}: V \rightarrow \mathcal{P}(V)$  is surjective, we now show that every such function is the outgoing map of a suitable edge relation  $E \subseteq V^2$ .

**Proposition 3 (Every function  $V \rightarrow \mathcal{P}(V)$  is an outgoing map):**

*For every set  $S$  and every function  $\mathcal{F}: S \rightarrow \mathcal{P}(S)$ , there is a binary relation  $E \subseteq S^2$  such that in the directed graph  $(S, E)$ , we have  $\mathcal{F}(s) = \text{Out}(s)$  for every  $s \in S$ .*

**Proof:** Put  $E = \{(s, t) \in S^2 \mid t \in \mathcal{F}(s)\}$ .

### 2.2.2. Alternative quasi-diagonal arguments

There are some other variants of the diagonal proof. For a positive integer  $n$ , let  $\mathcal{D}_n$  be the set of all the vertices that are not connected to themselves by an  $E$ -path of length  $n+1$ , i.e.

$$\mathcal{D}_n = \{v \in V \mid \text{there are no } \{w_i\}_{i=1}^n \text{ in } V \text{ such that } vEw_1E \dots Ew_nEv\}.$$

Let  $\mathcal{D}_\infty$  be the set of all the vertices for which there is no infinite  $E$ -path starting from them; in other words,

$$\mathcal{D}_\infty = \{v \in V \mid \text{there are no } \{w_i\}_{i=1}^\infty \text{ in } V \text{ such that } vEw_1E \dots Ew_nE \dots\}.$$

The proofs are similar to Cantor's [2]; the idea of  $\mathcal{D}_n$  is from Quine [3, §24], and  $\mathcal{D}_\infty$  is Raja's [4]. The following argument seems to

be new (see [5, 6] for some other proofs of Cantor’s power set theorem). Let  $\mathcal{S}$  be a fixed, nonempty set of natural numbers; that is,  $\emptyset \neq \mathcal{S} \subseteq \{0, 1, 2, 3, \dots\}$ . Let  $\mathcal{D}_{\mathcal{S}}$  be the set of all vertices that are not connected to themselves by an  $E$ -path of length  $n+1$  for any  $n \in \mathcal{S}$ , i.e.

$$\mathcal{D}_{\mathcal{S}} = \{v \in V \mid \neg vEv \text{ if } 0 \in \mathcal{S}, \text{ and for each } n \text{ with } 0 < n \in \mathcal{S}, \\ \text{there are no } \{w_i\}_{i=1}^n \text{ in } V \text{ such that } vEw_1E \dots Ew_nEv\}.$$

**Theorem 4 ( $\mathcal{D}_n, \mathcal{D}_{\infty}$ , and  $\mathcal{D}_{\mathcal{S}}$  are not outgoing sets):** *For any positive  $n \in \mathbb{N}$  and any nonempty subset  $\mathcal{S} \subseteq \mathbb{N}$ , none of the sets  $\mathcal{D}_n, \mathcal{D}_{\infty}$ , or  $\mathcal{D}_{\mathcal{S}}$  can be the outgoing set of a vertex  $v \in V$  in a directed graph  $(V, E)$ .*

**Proof:** We show that for each  $\chi \in \{n, \infty, \mathcal{S}\}$ , the symmetric difference  $\mathcal{D}_{\chi} \Delta \text{Out}(v) = [\text{Out}(v) \setminus \mathcal{D}_{\chi}] \cup [\mathcal{D}_{\chi} \setminus \text{Out}(v)]$  is nonempty. For that, we show the existence of some  $w_1 \in V$  such that  $\{v, w_1\} \cap [\mathcal{D}_{\chi} \Delta \text{Out}(v)] \neq \emptyset$ . In other words, either  $v$  is in the symmetric difference of  $\mathcal{D}_{\chi}$  and  $\text{Out}(v)$ , or  $w_1$  is in it.

- (1) If  $vEv$ , then  $v \in \text{Out}(v)$ , and the sequence  $vEvEv \dots$  can be continued indefinitely. So, for each  $\chi \in \{n, \infty, \mathcal{S}\}$ , we have  $v \notin \mathcal{D}_{\chi}$  by the definition of  $\mathcal{D}_{\chi}$ . Thus,  $v \in \text{Out}(v) \setminus \mathcal{D}_{\chi}$ , so  $v \in \mathcal{D}_{\chi} \Delta \text{Out}(v)$ .
- (2) Assume  $\neg vEv$ . Thus,  $v \notin \text{Out}(v)$ .
  - (2.i) If  $v \in \mathcal{D}_{\chi}$ , then  $v \in \mathcal{D}_{\chi} \setminus \text{Out}(v)$ , so  $v \in \mathcal{D}_{\chi} \Delta \text{Out}(v)$ .
  - (2.ii) Assume  $v \notin \mathcal{D}_{\chi}$ .
    - $(\chi = \infty)$ : By  $v \notin \mathcal{D}_{\infty}$ , there are  $w_1, w_2, \dots$  in  $V$  such that  $vEw_1Ew_2 \dots$ . But then  $w_1 \in \text{Out}(v)$  and  $w_1 \notin \mathcal{D}_{\infty}$ . Thus,  $w_1 \in \text{Out}(v) \setminus \mathcal{D}_{\infty}$ , so  $w_1 \in \mathcal{D}_{\infty} \Delta \text{Out}(v)$ .
    - $(\chi = n)$ : By  $v \notin \mathcal{D}_n$ , there exist some  $w_1, \dots, w_n$  in  $V$  such that  $vEw_1E \dots Ew_nEv$ . But then we have  $w_1 \in \text{Out}(v)$  and  $w_1 \notin \mathcal{D}_n$  by  $w_1E \dots Ew_nEvEw_1$ . Thus,  $w_1 \in \text{Out}(v) \setminus \mathcal{D}_n$ , so  $w_1 \in \mathcal{D}_n \Delta \text{Out}(v)$ .
    - $(\chi = \mathcal{S})$ : By  $v \notin \mathcal{D}_{\mathcal{S}}$ , either  $vEv$  or there are  $w_1, \dots, w_n \in V$  for some  $0 < n \in \mathcal{S}$  such that  $vEw_1E \dots Ew_nEv$ . The first option is in contradiction with our assumption  $\neg vEv$ . The second option implies  $w_1 \in \text{Out}(v)$  and  $w_1 \notin \mathcal{D}_{\mathcal{S}}$ , similarly to the case

of  $\chi = n$ . Thus,  $w_1 \in \text{Out}(v) \setminus \mathcal{D}_S$ , and so  $w_1 \in \mathcal{D}_S \Delta \text{Out}(v)$ .

Let us notice that  $\mathcal{D}_\infty \subseteq \mathcal{D}_n \subseteq \mathcal{D}$  for every  $n > 0$ , and that  $\mathcal{D}_S = \bigcap_{n \in S} \mathcal{D}_n$  with the convention  $\mathcal{D}_0 = \mathcal{D}$ .

## Note

1. Various internet misinformation pages and pseudoscientific forums are full of rejections and objections to this little argument.

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## ORCID

Saeed Salehi  <http://orcid.org/0000-0001-6961-5749>

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