A Non-Constructive Proof of Cantor's Theorem

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Cantor showed that there are *hierarchies of infinities*, not just *one infinity*; this gave a new life to the concept of infinity by bringing it into the domain of mathematics and freeing it from philosophers' dominance.

THEOREM (Georg Cantor, 1891). *There can be no surjection from a set A onto its powerset $\mathcal{P}(A)$.*

Cantor's original proof uses his celebrated diagonal argument, showing that for a function $f\colon A \to \mathcal{P}(A)$, the anti-diagonal set $B_0 = \{a \in A \mid a \notin f(a)\}$ is not in the range of f. Some alternative proofs are constructive, like Cantor's, as they explicitly construct some sets that are not in the range of f: If the binary relation $\mathcal{R} \subseteq A^2$ is defined by $x\mathcal{R}y$ if and only if $y \in f(x)$, for $x, y \in A$, and $n > 0$ is an integer, the set $B_n = \{a \in A \mid \nexists \{x_i\}_{i=1}^{n} : a\mathcal{R}x_1\mathcal{R}\cdots\mathcal{R}x_n\mathcal{R}a\}$ is out of the f's range (see [1, §24]); so is the set $B_\infty = \{a \in A \mid \nexists \{x_i\}_{i=1}^{\infty} : a\mathcal{R}x_1\mathcal{R}x_2\mathcal{R}\cdots\}$ (see [2]). There is a non-constructive proof that shows the nonexistence of an injection $h\colon \mathcal{P}(A) \to A$ (see [3]); a part of the proof is constructive as it builds two subsets $C, D \subseteq A$ such that $h(C) = h(D)$ but $C \neq D$ (one can even have $C \subsetneq D$ and $h(C) = h(D) \in D \setminus C$, see [4, §3]). However, that this implies the nonexistence of a surjection $f\colon A \to \mathcal{P}(A)$ requires the Axiom of Choice, thus making the whole argument non-constructive. We give another non-constructive proof (which seems new as it is not listed in the "various proofs" of [5]).

Proof. For finite sets, this follows from the Pigeonhole principle: if A has n elements, then prove by induction (on n) that $\mathcal{P}(A)$ has 2^n elements and that $2^n > n$ holds. If A is not finite, then partition it into some finite subsets, such as $A = \bigcup_{i \in I} A_i$, where the A_is, for $i \in I$, are nonempty and pairwise disjoint. We show that for a given function $f\colon A \to \mathcal{P}(A)$, there exists some $B \subseteq A$ that is out of f's range without explicitly describing what it could look like. For each $i \in I$, consider $f_i\colon A_i \to \mathcal{P}(A_i)$, defined by $f_i(x) = f(x) \cap A_i$, for $x \in A_i$. There exists a subset $B_i \subseteq A_i$ that is not in the range of f_i; recall that each A_i is finite. We show that the set $B = \bigcup_{i \in I} B_i$ is not in the range of f. If, otherwise, $B = f(\alpha)$ holds for some $\alpha \in A$, then there is a unique $\kappa \in I$ such that $\alpha \in A_\kappa$. Now, we have $B_\kappa = B \cap A_\kappa = f(\alpha) \cap A_\kappa = f_\kappa(\alpha)$, but this contradicts the choice of B_κ (which was supposed to be out of the f_κ's range). ∎

Let us notice the use of the Axiom of Choice in the above proof, once in partitioning the set A into $\{A_i\}_{i \in I}$ and once in *choosing* the subsets $\{B_i\}_{i \in I}$ of A_is. If one takes the A_is to be singletons, then one gets Cantor's set $B_0 = \bigcup_{a \in A}[\{a\} \setminus f(a)]$, since the only subset of the singleton $\{a\}$ that is not in the range of $g\colon \{a\} \to \mathcal{P}(\{a\})$ is the set $\{a\} \setminus g(a)$. A non-constructive proof is obtained, when the finite subsets A_is of A are taken to have more than one element.

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Summary. We offer a new proof (and review the known proofs) of Cantor's Powerset Theorem (1891), which concerns the non-existence of a surjective function from a set onto its powerset.

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